AREA UNDER CURVE

1. Let f : [0, 1] → [0, 1] be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region S = [0, 1]×[0,1]. Let G = {(x, y)∈S : y > f(x)} be called the green region and R ={(x, y)∈S : y < f(x)} be called the red region. Let L_h = {(x, h)∈S : x∈[0, 1]} be the horizontal line drawn at a height h ∈[0, 1]. Then which of the following statements is(are) ture?

[JEE(Advanced) 2023]

(A) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h

(B) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h

(C) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h

(D) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

2. Let n ≥ 2 be a natural number and f : [0,1] → ℝ be the function defined by **[JEE(Advanced) 2023]**

$$f(x) = \begin{cases} n(1-2nx) & \text{if } 0 \le x \le \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \le x \le \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \le x \le \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

If n is such that the area of the region bounded by the curves x = 0, x = 1, y = 0 and y = f (x) is 4, then the maximum value of the function f is

3. Consider the functions f, g : ℝ → ℝ defined by

$$f(x) = x^2 + \frac{5}{12} \text{ and } g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right), & |x| \le \frac{3}{4}, \\ 0, & |x| > \frac{3}{4}. \end{cases}$$

If α is the area of the region

$$\left\{(x,y) \in \mathbb{R} \times \mathbb{R} : |x| \le \frac{3}{4},\ 0 \le y \le \min\{f(x), g(x)\}\right\},$$

then the value of 9α is ______. **[JEE(Advanced) 2022]**

4. The area of the region $\left\{(x,y) : 0 \le x \le \frac{9}{4},\quad 0 \le y \le 1,\quad x \ge 3y,\quad x + y \ge 2\right\}$ is

[JEE(Advanced) 2021]

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

5. Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

[JEE(Advanced) 2020]

(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$ (B) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e-e^{-1})$

(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$ (D) $\left(2+\sqrt{3}\right)+\frac{1}{2}(e+e^{-1})$

6. The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is **[JEE(Advanced) 2019]**

(A) $8\log_e 2 - \frac{14}{3}$ (B) $16\log_e 2 - \frac{14}{3}$

(C) $16\log_e 2 - 6$ (D) $8\log_e 2 - \frac{7}{3}$

7. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$

for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE ?

[JEE(Advanced) 2018]

(A) The curve $y = f(x)$ passes through the point (1, 2)

(B) The curve $y = f(x)$ passes through the point (2, –1)

(C) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

8. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ______. **[JEE(Advanced) 2018]**

9. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x,\ 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced) 2017]**

(A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \le \frac{1}{2}$ (D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

10. Area of the region $\left\{(x,y) \in \mathbb{R}^2 : y \ge \sqrt{|x+3|}, 5y \le x + 9 \le 15\right\}$ is equal to - **[JEE(Advanced) 2016]**

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

11. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if F'(a) + 2 is the area of the region bounded by x = 0, y = 0, $y = f(x)$ and x = a, then $f(0)$ is

[JEE(Advanced) 2015]

SOLUTIONS

1. Ans. (B, C, D)

Sol. $f(x)=\frac{x^3}{3}-x^2+\frac{5x}{9}+\frac{17}{36}$

$f'(x)=x^2-2x+\frac{5}{9}$

$f'(x)=0$ at $x=\frac{1}{3}$ in [0, 1]

A_R = Area of Red region

A_G = Area of Green region

$A_R=\int_0^1 f(x)dx=\frac{1}{2}$

Total area = 1

$\Rightarrow A_G=\frac{1}{2}$

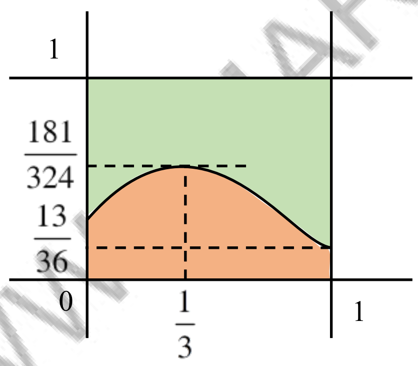
$\int_0^1 f(x)dx=\frac{1}{2}$

$A_G = A_R$

$f(0)=\frac{17}{36}$

$f(1)=\frac{13}{36}\approx 0.36$

$f\left(\frac{1}{3}\right)=\frac{181}{324}\approx 0.558$



(A) Correct when $h=\frac{3}{4}$ but $h\in\left[\frac{1}{4},\frac{2}{3}\right]$

$\Rightarrow$ (A) is incorrect

(B) Correct when $h=\frac{1}{4}$

$\Rightarrow$ (B) is correct

(C) When $h=\frac{181}{324}, A_R=\frac{1}{2}, A_G<\frac{1}{2}$

$h=\frac{13}{36}, A_R<\frac{1}{2}, A_G=\frac{1}{2}$

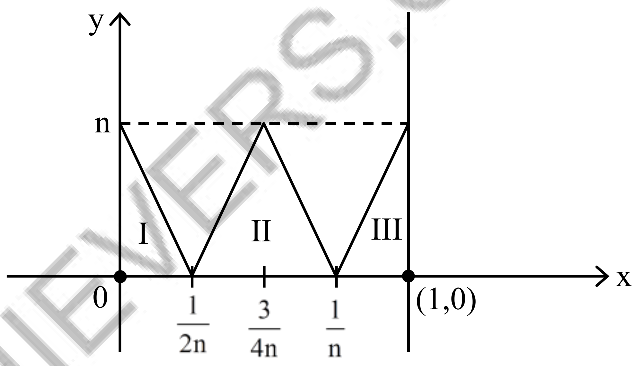
$\Rightarrow A_R = A_G$ for some $h\in\left(\frac{13}{36},\frac{181}{324}\right)$

$\Rightarrow$ (C) is correct

Option (D) is remaining coloured part of option (C), hence option (D) is also correct.

2. Ans. (8)

Sol.



Area = Area of (I + II + III) = 4

$=\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\left(1-\frac{1}{n}\right)\times n$

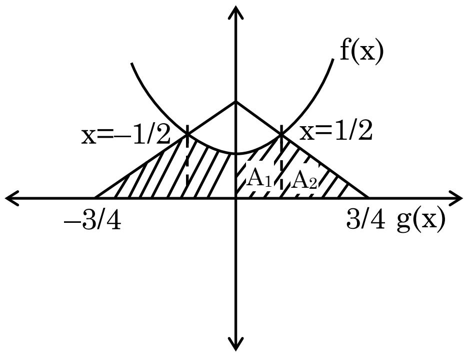
$=\frac{1}{4}+\frac{1}{4}+\frac{n-1}{2}=4$

$\boxed{n=8}$

$\therefore$ maximum value of f(x) = 8

3. Ans. (6)

Sol.



$x^2+\frac{5}{12}=\frac{2-8x}{3}$

$x^2+\frac{8x}{3}+\frac{5}{12}-2=0$

$12x^2 + 32x - 19 = 0$

$12x^2 + 38x - 6x - 19 = 0$

$2x(6x + 19) - 1(6x + 19) = 0$

$(6x + 19)(2x - 1) = 0$

$\boxed{x = \frac{1}{2}}$

$\alpha = 2(A_1 + A_2)$

$$\alpha = 2\left(\int_0^{1/2} x^2 + \frac{5}{12}dx + \frac{1}{2}\times\frac{1}{4}\times\frac{2}{3}\right)$$

$$\Rightarrow \alpha = 2\left[\left(\frac{x^3}{3} + \frac{5x}{12}\right)_0^{1/2} + \frac{1}{12}\right]$$

$$\Rightarrow \alpha = 2\left[\frac{1}{24} + \frac{5}{24} + \frac{1}{12}\right]$$

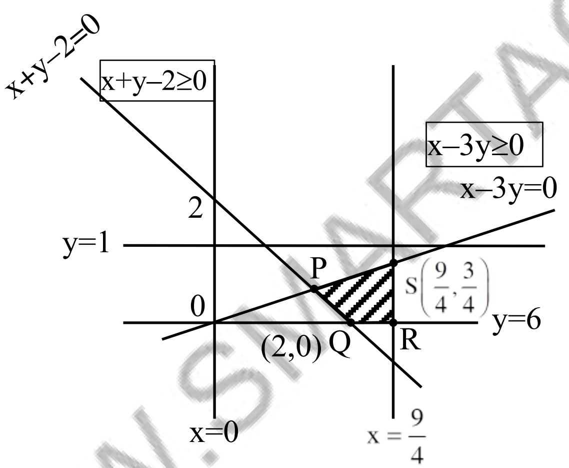
$$\Rightarrow \alpha = 2\left[\frac{1+5+2}{24}\right] \Rightarrow \alpha = 2\times\frac{8}{24} \Rightarrow 9\alpha = 9\times\frac{8}{12}$$

$$\Rightarrow 9\alpha = 6$$

4. Ans. (A)

Sol. $x + y - 2 = 0$

$P\left(\frac{3}{2}, \frac{1}{2}\right)$; $Q(2, 0)$; $R\left(\frac{9}{4}, 0\right)$; $S\left(\frac{9}{4}, \frac{3}{4}\right)$



$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ \frac{9}{4} & 0 \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & 0 \\ \frac{9}{4} & \frac{3}{4} \end{vmatrix} + \begin{vmatrix} \frac{9}{4} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} \end{vmatrix}\right|$$

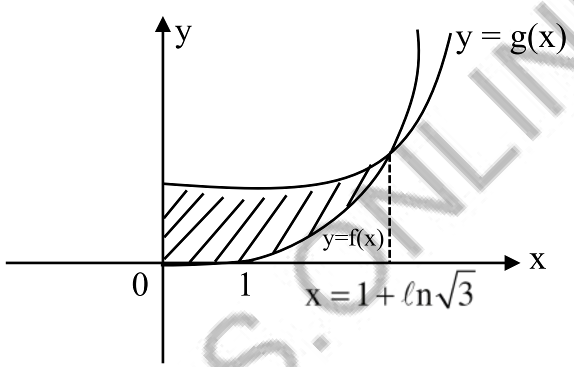
$$= \frac{1}{2}\left|(0-1) + (0-0) + \left(\frac{27}{16} - 0\right) + \left(\frac{9}{8} - \frac{9}{8}\right)\right|$$

$$= \frac{11}{32}$$

5. Ans. (A)

Sol. Here, $f(x) = \begin{cases} 0 & x \le 1 \\ e^{x-1} - e^{1-x} & x \ge 1 \end{cases}$

& $g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$



solve $f(x)$ & $g(x) \Rightarrow x = 1 + \ell n\sqrt{3}$

So bounded area $= \int_0^1 \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)dx +$

$$\int_1^{1+\ell n\sqrt{3}}\left[\frac{1}{2}\left(e^{x-1} + e^{1-x}\right) - \left(e^{x-1} - e^{1-x}\right)\right]dx$$

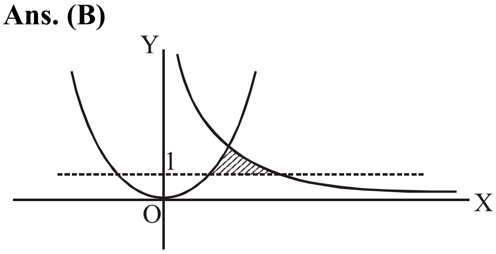
$$= \frac{1}{2}\left[e^{x-1} - e^{1-x}\right]_0^1 + \left[-\frac{1}{2}e^{x-1} - \frac{3}{2}e^{1-x}\right]_1^{1+\ell n\sqrt{3}}$$

$$= \frac{1}{2}\left[e - \frac{1}{e}\right] + \left[\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + 2\right]$$

$$= 2 - \sqrt{3} + \frac{1}{2}\left(e - \frac{1}{e}\right)$$

6. Ans. (B)

Sol.



For intersection, $\frac{8}{y} = \sqrt{y} \Rightarrow y = 4$

Hence, required area $= \int_1^4\left(\frac{8}{y} - \sqrt{y}\right)dy$

$$= \left[8\ell ny - \frac{2}{3}y^{3/2}\right]_1^4 = 16\ell n2 - \frac{14}{3}$$

Remark : The question should contain the phrase "area of the bounded region in the first quadrant". Because, in the 2^{nd} quadrant, the region above the line $y = 1$ and below $y = x^2$, satisfies the region, which is unbounded.

7. Ans. (B, C)

Sol. $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$

$\Rightarrow \quad e^{-x} f(x) = e^{-x}(1-2x) + \int_0^x e^{-t} f(t)\,dt$

Differentiate w.r.t. x.

$-e^{-x} f(x) + e^{-x} f'(x)$

$= -e^{-x}(1-2x) + e^{-x}(-2) + e^{-x} f(x)$

$\Rightarrow \quad -f(x) + f'(x) = -(1-2x) - 2 + f(x).$

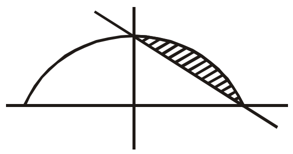
$\Rightarrow \quad f'(x) - 2f(x) = 2x - 3$

Integrating factor $= e^{-2x}$.

$f(x).e^{-2x} = \int e^{-2x}(2x-3)\,dx$

$= (2x-3)\int e^{-2x}dx - \int\left((2)\int e^{-2x}dx\right)dx$

$= \dfrac{(2x-3)e^{-2x}}{-2} - \dfrac{e^{-2x}}{2} + c$



$f(x) = \dfrac{2x-3}{-2} - \dfrac{1}{2} + ce^{2x}$

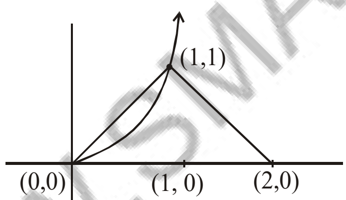
$f(0) = \dfrac{3}{2} - \dfrac{1}{2} + c = 1 \Rightarrow c = 0$

$\therefore \quad f(x) = 1 - x$

Area $= \dfrac{\pi}{4} - \dfrac{1}{2} = \dfrac{\pi - 2}{4}$

8. Ans. (4)

Sol.



Area $= \int_0^1 (x - x^n)\,dx = \dfrac{3}{10}$

$\left[\dfrac{x^2}{2} - \dfrac{x^{n+1}}{n+1}\right]_0^1 = \dfrac{3}{10}$

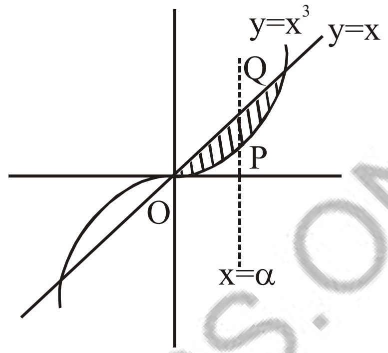
$\dfrac{1}{2} - \dfrac{1}{n+1} = \dfrac{3}{10} \qquad \therefore\ n + 1 = 5$

$\Rightarrow n = 4$

9. Ans. (A, D)

Sol. Area between $y = x^3$ and $y = x$ in $x \in (0,1)$ is

$A = \int_0^1 \left(x - x^3\right)dx = \dfrac{1}{4}$



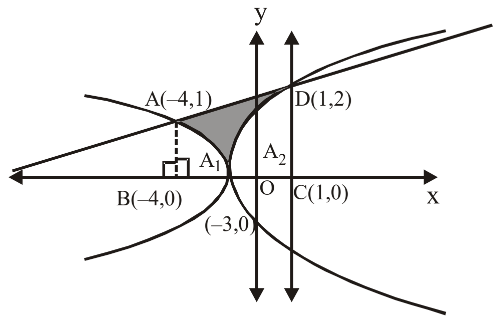
Area of curve linear triangle OPQ $= \dfrac{A}{2} = \dfrac{1}{8}$

$\Rightarrow \int_0^\alpha \left(x - x^3\right)dx = \dfrac{1}{8} \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$

$\Rightarrow \left(\alpha^2 - 1\right)^2 = \dfrac{1}{2} \Rightarrow \alpha^2 = \dfrac{\sqrt{2}-1}{\sqrt{2}}$

10. Ans. (C)

Sol.



Clearly required area

= area (trapezium ABCD) – ($A_1 + A_2$) ....(i)

area (trapezium ABCD) $= \dfrac{1}{2}(1+2)(5) = \dfrac{15}{2}$

$A_1 = \int_{-4}^{-3} \sqrt{-(x+3)}\,dx = \dfrac{2}{3}$

and $A_2 = \int_{-3}^{1} (x+3)^{1/2}dx = \dfrac{16}{3}$

$\therefore$ From equation (1), we get required area

$= \dfrac{15}{2} - \left(\dfrac{2}{3} + \dfrac{16}{3}\right) = \dfrac{3}{2}$

11. Ans. (3)

Sol. From the question

$$\int_0^a f(x) = F'(a) + 2$$

Differentiating, we get

$$f(a) = F''(a) \Rightarrow f(0) = F''(0)$$

Now, $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$,

$$\therefore\ F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right) \times 2x - 2\left(\cos^2 x\right)$$

$$F''(x) = 4\left(\cos^2\left(x^2 + \frac{\pi}{6}\right) - 4x^2 \cos\right.$$

$$\left.\left(x^2 + \frac{\pi}{6}\right)\sin\left(x^2 + \frac{\pi}{6}\right)\right) + 4\cos x \sin x$$

$$F''(0) = 4\cos^2\frac{\pi}{6} = 3$$

$\therefore\ f(0) = 3$