DEFINITE INTEGRATION

1. Let $f : (0, 1) \to \mathbb{R}$ be the functions defined as $f(x) = \sqrt{n}$ if $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ where $n \in N$. Let $g : (0, 1) \to \mathbb{R}$ be a function such that $\int_{x^2}^{x} \sqrt{\frac{1-t}{t}}\,dt < g(x) < 2\sqrt{x}$ for all $x \in (0, 1)$. Then $\lim_{x\to 0} f(x)g(x)$

[JEE(Advanced) 2023]

(A) does **NOT** exist

(B) is equal to 1

(C) is equal to 2

(D) is equal to 3

2. For $x \in \mathbb{R}$, let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the minimum value of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \int_0^{x\tan^{-1}x} \frac{e^{(t-\cos t)}}{1+t^{2023}}\,dt$ is **[JEE(Advanced) 2023]**

3. Consider the equation

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}\,dx = 1,\quad a \in (-\infty, 0) \cup (1, \infty).$$

Which of the following statements is/are TRUE ? **[JEE(Advanced) 2022]**

(A) **No** a satisfies the above equation

(B) An integer a satisfies the above equation

(C) An irrational number a satisfies the above equation

(D) More than one a satisfy the above equation

4. The greatest integer less than or equal to

$$\int_1^2 \log_2(x^3+1)\,dx + \int_1^{\log_2 9} \left(2^x - 1\right)^{\frac{1}{3}}\,dx$$

is______. **[JEE(Advanced) 2022]**

5. For positive integer n, define

$$f(n) = n + \frac{16+5n-3n^2}{4n+3n^2} + \frac{32+n-3n^2}{8n+3n^2} + \frac{48-3n-3n^2}{12n+3n^2} + \ldots + \frac{25n-7n^2}{7n^2}.$$

Then, the value of $\lim_{n\to\infty} f(n)$ is equal to **[JEE(Advanced) 2022]**

(A) $3 + \frac{4}{3}\log_e 7$

(B) $4 - \frac{3}{4}\log_e\left(\frac{7}{3}\right)$

(C) $4 - \frac{4}{3}\log_e\left(\frac{7}{3}\right)$

(D) $3 + \frac{3}{4}\log_e 7$

6. Let $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ be a continuous function such that

$$f(0)=1 \text{ and } \int_0^{\frac{\pi}{3}} f(t)\,dt=0$$

Then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(B) The equation $f(x)-3\sin 3x=-\frac{6}{\pi}$ has at least one solution in $\left(0,\frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\lim\limits_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Question Stem for Questions Nos. 7 and 8

Question Stem

Let $g_i:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$, i = 1, 2, and $f:\left[\frac{\pi}{8},\frac{3\pi}{8}\right]\to\mathbb{R}$ be functions such that

$g_1(x)=1$, $g_2(x)=|4x-\pi|$ and $f(x)=\sin^2 x$, for all $x\in\left[\frac{\pi}{8},\frac{3\pi}{8}\right]$

Define $S_i=\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x)\cdot g_i(x)dx$, $i=1, 2$

7. The value of $\frac{16S_1}{\pi}$ is _____. **[JEE(Advanced) 2021]**

8. The value of $\frac{48S_2}{\pi^2}$ is _____. **[JEE(Advanced) 2021]**

Paragraph for Question No. 9 and 10

Let $\psi_1:[0,\infty)\to\mathbb{R}$, $\psi_2:[0,\infty)\to\mathbb{R}$, $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ be functions such that

$f(0)=g(0)=0$,

$\psi_1(x)=e^{-x}+x,\quad x\ge 0,$

$\psi_2(x)=x^2-2x-2e^{-x}+2,\ x\ge 0,$

$f(x)=\int_{-x}^{x}\left(|t|-t^2\right)e^{-t^2}\,dt,\ x>0$

and

$g(x)=\int_0^{x^2}\sqrt{t}\,e^{-t}\,dt,\ x>0$

9. Which of the following statements is **TRUE** ? **[JEE(Advanced) 2021]**

(A) $f\left(\sqrt{\ln 3}\right)+g\left(\sqrt{\ln 3}\right)=\frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

10. Which of the following statements is **TRUE** ? **[JEE(Advanced) 2021]**

(A) $\psi_1(x) \le 1$, for all $x > 0$

(B) $\psi_2(x) \le 0$, for all $x > 0$

(C) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

11. For any real number x, let [x] denote the largest integer less than or equal to x. If then the value of 9I is _______. **[JEE(Advanced) 2021]**

12. Which of the following inequalities is/are TRUE? **[JEE(Advanced) 2020]**

(A) $\int_0^1 x\cos x\,dx \ge \frac{3}{8}$ (B) $\int_0^1 x\sin x\,dx \ge \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\,dx \ge \frac{1}{2}$ (D) $\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F : [0, \pi] \to \mathbb{R}$ is defined by $F(x) = \int_0^x f(t)dt$, and if

$$\int_0^{\pi} (f'(x) + F(x))\cos x\,dx = 2,$$

then the value of $f(0)$ is _____. **[JEE(Advanced) 2020]**

14. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals _____. **[JEE(Advanced) 2019]**

15. For $a \in \mathbb{R}$, $|a| > 1$, let $\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.......+\frac{1}{(an+n)^2}\right)}\right)=54$. Then the possible value(s) of a is/are : **[JEE(Advanced) 2019]**

(A) 8 (B) –9 (C) –6 (D) 7

16. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$ equals **[JEE(Advanced) 2019]**

17. For each positive integer n, let $y_n = \frac{1}{n}(n+1)(n+2)...(n+n)^{1/n}$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is ______. **[JEE(Advanced) 2018]**

18. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}} dx$$

is ______. **[JEE(Advanced) 2018]**

19. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right) = 3$ and f'(0) = 1. If

$$g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\,\text{cosec}\, t - \cot t\,\text{cosec}\, t\, f(t)]\, dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$ **[JEE(Advanced) 2017]**

20. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then **[JEE(Advanced)-2017]**

(A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

21. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$, then **[JEE(Advanced) 2017]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$

(C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

22. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to **[JEE(Advanced) 2016]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

23. Let $f(x) = \lim_{n\to\infty} \left(\frac{n^n (x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)} \right)^{x/n}$, for all x > 0. Then **[JEE(Advanced) 2016]**

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$ (C) $f'(2) \le 0$ (D) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

24. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is

[JEE(Advanced) 2016]

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x] & , \quad x \le 2 \\ 0 & , \quad x > 2 \end{cases}$,

where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is **[JEE(Advanced) 2015]**

26. If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE(Advanced) 2015]**

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1,2]$ and $G(x) = \int_{-1}^{x} t\left|f\left(f(t)\right)\right|dt$ for all $x \in [-1,2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE(Advanced) 2015]**

28. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L\ ?$$

[JEE(Advanced) 2015]

(A) $a = 2,\ L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) $a = 2,\ L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a = 4,\ L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) $a = 4,\ L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

29. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

[JEE(Advanced) 2015]

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

30. Let $f'(x) = \dfrac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\dfrac{1}{2}\right) = 0$. If $m \le \displaystyle\int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are **[JEE(Advanced) 2015]**

(A) $m = 13, M = 24$

(B) $m = \dfrac{1}{4}, M = \dfrac{1}{2}$

(C) $m = -11, M = 0$

(D) $m = 1, M = 12$

Paragraph For Questions 31 and 32

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$, $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

31. The correct statement(s) is(are) **[JEE(Advanced) 2015]**

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \ne 0$ for any $x \in (1, 3)$

(D) $f'(x) = 0$ for some $x \in (1, 3)$

32. If $\displaystyle\int_1^3 x^2F'(x)\,dx = -12$ and $\displaystyle\int_1^3 x^3F''(x)\,dx = 40$, then the correct expression(s) is(are) **[JEE(Advanced) 2015]**

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\displaystyle\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\displaystyle\int_1^3 f(x)dx = -12$

33. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : \mathbb{R} \to \mathbb{R}$ be defined as **[JEE(Advanced) 2014]**

$$g(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \int_a^x f(t)\,dt & \text{if} \quad a \le x \le b \\ \int_a^b f(t)\,dt & \text{if} \quad x > b \end{cases}.$$

Then

(A) g(x) is continuous but not differentiable at a

(B) g(x) is differentiable on $\mathbb{R}$

(C) g(x) is continuous but not differentiable at b

(D) g(x) is continuous and differentiable at either a or b but not both.

34. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = \displaystyle\int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then **[JEE(Advanced) 2014]**

(A) $f(x)$ is monotonically increasing on $[1, \infty)$

(B) $f(x)$ is monotonically decreasing on $[0, 1)$

(C) $f(x) + f\left(\dfrac{1}{x}\right) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on $\mathbb{R}$

35. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **[JEE(Advanced) 2014]**

36. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec}x)^{17}dx$ is equal to - **[JEE(Advanced) 2014]**

(A) $\int_0^{\log(1+\sqrt{2})}2(e^u+e^{-u})^{16}du$

(B) $\int_0^{\log(1+\sqrt{2})}(e^u+e^{-u})^{17}du$

(C) $\int_0^{\log(1+\sqrt{2})}(e^u-e^{-u})^{17}du$

(D) $\int_0^{\log(1+\sqrt{2})}2(e^u-e^{-u})^{16}du$

37. Let $f:[0,2]\to\mathbb{R}$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0)=1$. Let $F(x)=\int_0^{x^2} f(\sqrt{t})dt$ for $x\in[0,2]$. If $F'(x)=f'(x)$ for all $x\in(0,2)$, then F(2) equals -

[JEE(Advanced) 2014]

(A) e^2-1

(B) e^4-1

(C) $e-1$

(D) e^4

Paragraph For Questions Nos. 38 and 39

Given that for each $a\in(0,1)$, $\lim_{h\to0^+}\int_h^{1-h}t^{-a}\left(1-t\right)^{a-1}dt$ exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0,1).

38. The value of $g\left(\frac{1}{2}\right)$ is - **[JEE(Advanced) 2014]**

(A) π

(B) 2π

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{4}$

39. The value of $g\left(\frac{1}{2}\right)$ is- **[JEE(Advanced) 2014]**

(A) $\frac{\pi}{2}$

(B) π

(C) $-\frac{\pi}{2}$

(D) 0

40.

	List-I	List-II	
P.	The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying f(0) = 0 and $\int_0^1 f(x)dx = 1$, is	**1.**	8
Q.	The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.**	2
R.	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	**3.**	4
S.	$\frac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_0^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	**4.**	0

Codes : **[JEE(Advanced) 2014]**

	P	Q	R	S
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

SOLUTIONS

1. Ans. (C)

Sol. $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt.\sqrt{n}\le f(x)g(x)\le 2\sqrt{x}\sqrt{n}$

$\because \int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt=\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}$

$\Rightarrow \lim_{x\to 0}\left(\frac{\sin^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}-\sin^{-1}x-x\sqrt{1-x^2}}{\sqrt{x}}\le f(x)g(x)\le\frac{2\sqrt{x}}{\sqrt{x}}\right)$

$\Rightarrow 2\le \lim_{x\to 0}f(x)g(x)\le 2$

$\Rightarrow \lim_{x\to 0}f(x)g(x)=2$

2. Ans. (0)

Sol. $f(x)=\int_0^{x\tan^{-1}x}\frac{e^{t-\cos t}}{1+t^{2023}}dt$

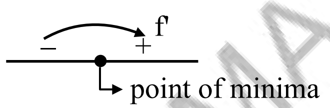
$f'(x)=\frac{e^{x\tan^{-1}x-\cos(x\tan^{-1}x)}}{1+(x\tan^{-1}x)^{2023}}\cdot\left(\frac{x}{1+x^2}+\tan^{-1}x\right)$

For $x<0,\ \tan^{-1}x\in\left(-\frac{\pi}{2},0\right)$

For $x\ge 0,\ \tan^{-1}x\in\left[0,\frac{\pi}{2}\right)$

$\Rightarrow x\tan^{-1}x\ge 0\ \forall\ x\in R$

And $\frac{x}{1+x^2}+\tan^{-1}x=\begin{cases}>0 & \text{For } x>0\\ <0 & \text{For } x<0\\ 0 & \text{For } x=0\end{cases}$



Hence minimum value is $f(0)=\int_0^0=0$

3. Ans. (C, D)

Sol. $\int_1^e\frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}=1$

Let $a-(\log_e x)^{3/2}=t$

$\frac{(\log_e x)^{1/2}}{x}dx=-\frac{2}{3}dt$

$=\frac{2}{3}\int_a^{a-1}\frac{-dt}{t^2}=\frac{2}{3}\left(\frac{1}{t}\right)_a^{a-1}=1$

$\frac{2}{3a(a-1)}=1$

$3a^2-3a-2=0$

$a=\frac{3\pm\sqrt{33}}{6}$

4. Ans. (5)

Sol. $f(x)=\log_2(x^3+1)=y$

$x^3+1=2^y\Rightarrow x=(2^y-1)^{1/3}=f^{-1}(y)$

$f^{-1}(x)=(2^x-1)^{1/3}$

$=\int_1^2\log_2(x^3+1)dx+\int_1^{\log_2 9}(2^x-1)^{1/3}dx$

$=\int_1^2 f(x)dx+\int_1^{\log_2 9}f^{-1}(x)dx=2\log_2 9-1$

$=8<9<2^{7/2}\Rightarrow 3<\log_2 9<\frac{7}{2}$

$=5<2\log_2 9-1<6$

$[2\log_2 9-1]=5$

5. Ans. (B)

Sol. $f(n)=n+\sum_{r=1}^{n}\frac{16r+(9-4r)n-3n^2}{4rn+3n^2}$

$f(n)=n+\sum_{r=1}^{n}\frac{(16r+9n)-(4rn+3n^2)}{4rn+3n^2}$

$f(n)=n+\left(\sum_{r=1}^{n}\frac{16r+9n}{4rn+3n^2}\right)-n$

$\lim_{n\to\infty}f(n)=\lim_{n\to\infty}\sum\frac{16r+9n}{4rn+3n^2}$

$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(16\left(\frac{r}{n}\right)+9\right)\frac{1}{n}}{4\left(\frac{r}{n}\right)+3}$

$=\int_0^1\frac{16x+9}{4x+3}dx=\int_0^1 4\,dx-\int_0^1\frac{3\,dx}{4x+3}$

$=4-\frac{3}{4}(\ell n|4x+3|)_0^1$

$=4-\frac{3}{4}\ell n\frac{7}{3}$

6. Ans. (A, B, C)

Sol. (A) Let $g(x) = f(x) - 3\cos 3x$

Now,

$$\int_0^{\pi/3} g(x)dx = \int_0^{\pi/3} f(x)dx - 3\int_0^{\pi/3} \cos 3x\, dx = 0$$

Hence $g(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

(B) Let $h(x) = f(x) - 3\sin 3x + \frac{6}{\pi}$

Now,

$$\int_0^{\pi/3} h(x)dx = \int_0^{\pi/3} f(x)dx - 3\int_0^{\pi/3} \sin 3x\, dx + \int_0^{\pi/3} \frac{6}{\pi}dx$$

$= 0 - 2 + 2 = 0$

Hence $h(x) = 0$ has a root in $\left(0, \frac{\pi}{3}\right)$

(C) $$\lim_{x\to 0}\frac{x\int_0^x f(t)\,dt}{1-e^{x^2}} = \lim_{x\to 0}\underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1}\underbrace{\frac{\int_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopital's Rule}}$$

$$= -1\lim_{x\to 0}\frac{f(x)}{1} = -1$$

(D) $$\lim_{x\to 0}\frac{(\sin x)\int_0^x f(t)\,dt}{x^2}$$

$$= \lim_{x\to 0}\underbrace{\left(\frac{\sin x}{x}\right)}_{1}\underbrace{\frac{\int_0^x f(t)\,dt}{x}}_{\text{Apply L' Hopitals Rule}}$$

$$= 1\lim_{x\to 0}\frac{f(x)}{1} = 1$$

7. Ans. (2.00)

Sol.

$$S_1 = \int_{\pi/8}^{3\pi} f(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x\, dx = \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx$$

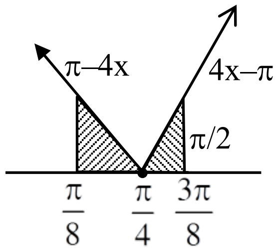
$$= \int_{\pi/8}^{3\pi/8} \cos^2 x dx$$

$$2S_1 = \int_{\pi/8}^{3\pi/8} (\sin^2 x + \cos^2 x)dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{\pi}{4}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

8. Ans. (1.50)

Sol.



$$S_2 = \int_{\pi/8}^{3\pi/8} f(x)g_2(x)dx = \int_{\pi/8}^{3\pi/8} \sin^2 x|4x-\pi|dx$$

$$= \int_{\pi/8}^{3\pi/8} \sin^2\left(\frac{\pi}{2}-x\right)\left|4\left(\frac{\pi}{2}-x\right)-\pi\right|dx$$

$$= \int_{\pi/8}^{3\pi/8} (\cos^2 x)|\pi - 4x|dx$$

$$\Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|\ (\sin^2 x + \cos^2 x)dx$$

$$= \int_{\pi/8}^{3\pi/8} |4x-\pi|dx$$

$$= 2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2} = 1.5$$

9. Ans. (C)

Sol. $f'(x) = (|x|-x^2)e^{-x^2} + (|x|-x^2)e^{-x^2},\ x \ge 0$

$f' = 2(x - x^2)\ e^{-x^2}$

$$\begin{array}{ccccc} & & +++ & & --- \\ \hline & 0 & & 1 & \end{array}$$

hence option (D) is wrong

$g'(x) = xe^{-x^2}\,2x$

$f'(x) + g'(x) = 2 \times e^{-x^2}$

$f(x) + g(x) = -e^{-x^2} + c$

$f(x) + g(x) = -e^{-x^2} + 1$

$f(\ell n3) + g\left(\sqrt{\ell n3}\right) = 1 - \frac{1}{3} = \frac{2}{3}$

(option (A) is wrong)

$H(x) = \psi_1(x) - 1 - \alpha x = e^{-x} + x - 1 - \alpha x$,

$x \ge 1\ \&\ \alpha \in (1, x)$

$H(1) = e^{-1} + 1 - 1 - \alpha < 0$

$H'(x) = -e^{-x} + 1 - \alpha > 0 \Rightarrow H(x)$ is $\downarrow$

$\Rightarrow$ option (B) is wrong

(C) $\psi_2(x) = 2(\psi_1(\beta) - 1)$

Applying L.M.V.T to $\psi_2(x)$ in [0, x]

$$\psi_2'(\beta) = \frac{\psi_2(x) - \psi_2(0)}{x}$$

$$2\beta - 2 + 2e^{-\beta} = \frac{\psi_2(x) - 0}{x}$$

$\Rightarrow \psi_2(x) = 2x(\psi_1(\beta) - 1)$ has one solution

option (C) is correct.

10. Ans. (D)

Sol. (A) $\psi_1(x) = e^{-x} + x,\ x \geq 0$

$\psi_1'(x) = 1 - e^{-x} > 0 \Rightarrow \psi_1(x)$ is $\uparrow$

$\psi_1(x) \geq \psi_1(0) \quad \forall x \geq 0 \Rightarrow \psi_1(x) \geq 1$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x} \quad x \geq 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \geq 0 \quad \forall x \geq 0$

$\Rightarrow \psi_2(x)$ is $\uparrow$

$\Rightarrow \psi_2(x) \geq \psi_2(0) \Rightarrow \psi_2(x) \geq 0$

(C) $f(x) = 2\int_0^x (t - t^2)e^{-t^2}dt$ & $x \in \left(0, \frac{1}{2}\right)$

Let, $H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5,\ x \in \left(0, \frac{1}{2}\right)$

$H(0) = 0$

$H'(x) = 2(x - x^2)e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$= -2x^2e^{-x^2} + 2x^2 - 2x^4$

$= 2x^2(1 - x^2 - e^{-x^2})$

$\because e^{-x} \geq 1 - x \ \forall x \geq 0$

$\Rightarrow H'(x) \leq 0$

$\Rightarrow H(x)$ is $\downarrow \Rightarrow H(x) \leq 0\ \forall x \in \left(0, \frac{1}{2}\right)$

$f(x) \leq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5; \forall x \in \left(0, \frac{1}{2}\right)$

Let $P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7 x \in \left(0, \frac{1}{2}\right)$

$P'(x) = 2x^2e^{-x^2} - 2x^2 + 2x^4 - x^6$

$= 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3} + \ldots\right) - 2x^2 + 2x^4 - x^6$

$= -\frac{x^8}{3} + \frac{x^{10}}{12}$..........

$\Rightarrow P'(x) \leq 0$

$\Rightarrow P(x)$ is $\downarrow$

$\Rightarrow P(x) \leq 0$

option (D) is correct

11. Ans. (182)

Sol. Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\frac{(x+1) - x}{(x+1)^2}\right) = \frac{10}{(x+1)^2} > 0$

$\forall x \in [0, 10]$,

So, $f(x)$ is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$I = \int_0^{1/9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx +$$

$$\int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$= 0 + \int_{1/9}^{2/3}dx + 2\int_{2/3}^{9}dx + 3\int_{9}^{10}dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5 + 150 + 27}{9} = \frac{182}{9} \Rightarrow 9I = 182$$

12. Ans. (A, B, D)

Sol. (A) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$\cos x \geq 1 - \frac{x^2}{2}$

$$\int_0^1 x\cos x \geq \int_0^1 x\left(1-\frac{x^2}{2}\right) = \frac{1}{2}-\frac{1}{8}$$

$$\int_0^1 x\cos x \geq \frac{3}{8} \quad \text{(True)}$$

(B) $\sin x \geq x - \frac{x^3}{6}$

$$\int_0^1 x\sin x \geq \int_0^1 x\left(x-\frac{x^3}{6}\right)dx$$

$$\int_0^1 x\sin x \geq \frac{1}{3}-\frac{1}{30} \Rightarrow \int_0^1 x\sin x\ dx \geq \frac{3}{10}$$

(True)

(D) $$\int_0^1 x^2\sin x\ dx \geq \int_0^1 x^2\left(x-\frac{x^3}{6}\right)dx$$

$$\int_0^1 x^2\sin x\ dx \geq \frac{1}{4}-\frac{1}{36}$$

$$\int_0^1 x^2\sin x\ dx \geq \frac{2}{9} \text{ (True)}$$

(C) $\cos x < 1$

$x^2 \cos x < x^2$

$$\int_0^1 x^2\cos x\ dx < \int_0^1 x^2 dx$$

$$\int_0^1 x^2\cos x\ dx < \frac{1}{3}$$

So, option (C) is incorrect.

13. Ans. (1080.00)

Sol. $F(x) = \int_0^x f(t).\ dt$

$\Rightarrow F'(x) = f(x)$

$$I = \int_0^\pi f'(x).\cos x\ dx + \int_0^\pi F(x)\cos(x)\ dx = 2 \quad ...(1)$$

$$I_1 = \int_0^\pi f'(x).\cos x\ dx \quad \text{(Let)}$$

Using by parts

$$I_1 = \left(\cos x.f(x)\right)_0^\pi + \int_0^\pi \sin x.f(x)dx$$

$$I_1 = 6 - f(0) + \int_0^\pi \sin x.F'(x)dx$$

$I_1 = 6 - f(0) + I_2$ ...(2)

$$I_2 = \int_0^\pi \sin x.F'(x).dx$$

Using by part we get

$$I_2 = (\sin x.F(x))_0^\pi - \int_0^\pi \cos x.F(x)\ dx$$

$$I_2 = -\int_0^\pi \cos x.F(x)\ dx$$

(2) $\Rightarrow I_1 = 6 - f(0) - \int_0^\pi \cos x.F(x)\ dx$

(1) $\Rightarrow I = 6 - f(0) = 2 \Rightarrow f(0) = 4$

14. Ans. (4.00)

Sol.

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{(1+e^{\sin x})(2-\cos 2x)} + \frac{1}{(1+e^{-\sin x})(2-\cos 2x)}\right]dx$$

(using King's Rule)

$$\Rightarrow \quad I = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$$

$$\Rightarrow \quad I = \frac{2}{\pi}\int_0^{\pi/4}\frac{dx}{2-\cos 2x} = \frac{2}{\pi}\int_0^{\pi/4}\frac{\sec^2 x dx}{1+3\tan^2 x}$$

$$= \frac{2}{\sqrt{3}\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/4} = \frac{2}{3\sqrt{3}}$$

$$\Rightarrow \quad 27I^2 = 27\times\frac{4}{27} = 4$$

15. Ans. (A, B)

Sol. $$\lim_{n\to\infty}\frac{n^{1/3}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)\right)^{1/3}}{n^{7/3}\left(\sum_{r=1}^{n}\frac{1}{(an+r)^2}\right)} = 54$$

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(a+r/n)^2}}\right) = 54$$

$$\Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{1}{(a+x)^2}dx} = 54$$

$$\Rightarrow \frac{\frac{3}{4}}{\frac{1}{a(a+1)}} = 54$$

$\Rightarrow$ a(a + 1) = 72

$\Rightarrow$ $a^2 + a - 72 = 0 \Rightarrow a = -9, 8$

16. Ans. (0.50)

Sol. $$I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$$

$$= \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}d\theta$$

$$2I = \int_0^{\pi/2} \frac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}$$

$$= 3\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta} = t$

$$\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta = dt$$

$\sec^2\theta d\theta = 2(t-1)dt$

$$= 3\int_1^{\infty} \frac{2(t-1)dt}{t^4}$$

$$= 6\int_1^{\infty}\left(t^{-3}-t^{-4}\right)dt$$

$$2I = 6\left(\frac{t^{-2}}{-2}-\frac{t^{-3}}{-3}\right)_1^{\infty} = 6\left[0-0-\left\{-\frac{1}{2}+\frac{1}{3}\right\}\right]$$

I = 0.50

17. Ans. (1)

Sol. $$y_n = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$y_n = \prod_{r=1}^{n}\left(1+\frac{r}{n}\right)^{1/n}$$

$$\log y_n = \frac{1}{n}\sum_{r=1}^{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log y_n = \lim_{x\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \ell n(1+x)dx$$

$$\Rightarrow \log L = \log\frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e}$$

$\Rightarrow$ [L] = 1

18. Ans. (2)

Sol. $$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put $\frac{1-x}{1+x} = t \Rightarrow = \frac{-2dx}{(1+x)^2}dt$

$$I = \int_1^{1/3}\frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2}\times\left|\frac{-2}{\sqrt{t}}\right|_1^{1/3}$$

$$= \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$$

19. Ans. (2)

Sol. $$g(x) = \int_x^{\pi/2}(f'(t)\operatorname{cosec}t - f(t)\operatorname{cosec}t\cot t)dt$$

$$= \int_x^{\pi/2}\left(f(t)\operatorname{cosec}t\right)'dt$$

$$= f\left(\frac{\pi}{2}\right)\operatorname{cosec}\left(\frac{\pi}{2}\right) - \frac{f(x)}{\sin x} = 3 - \frac{f(x)}{\sin x}$$

$\therefore$ $\lim_{x\to 0}g(x) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$; as $f'(0) = 1$

$$\Rightarrow \lim_{x\to 0}g(x) = 3-1 = 2$$

20. Ans. (B,C)

Sol. $S_k = \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx = \sum_{k=1}^{98} I_k$

$1 \le k \le x \le k+1 \le 99$

$2 \le k+1 \le x+1 \le k+2 \le 100$

$$\frac{k+1}{(k+1)(100)} \le \frac{k+1}{x(x+1)} \le \frac{k+1}{x(x+1)}$$

$$\int_k^{k+1}\frac{1}{100} \le \int_k^{k+1}\frac{k+1}{x[x+1)} \le \int_k^{k+1}\frac{1}{x}$$

$$\frac{1}{100} \le I_k \le \ell n\left(\frac{k+1}{k}\right)$$

$$\frac{98}{100} \le \sum_{4\alpha=1}^{98} I_k \le \ell n99$$

$$\frac{49}{50} \le S_k \le \ell n99$$

Aliter

$$I = \sum_{k=1}^{98}\left(\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx\right)$$

$$= \sum_{k=1}^{98}(k+1)\left(\int_k^{k+1}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx\right)$$

$$= \sum_{k=1}^{98}(k+1)\left((\ell n\, x - \ell n\,(x+1))_k^{k+1}\right)$$

$$= \sum_{k=1}^{98}(k+1)(\ell n\,(k+1) - \ell n\,(k+2) - \ell n\, k + \ell n\,(k+1))$$

$$= \sum_{k=1}^{98}((k+1)\ell n\,(k+1) - k.\ell n\, k) - \sum_{k=1}^{98}((k+1).\ell n\,(k+2) - k.\ell n\,(k+1)) + \sum_{k=1}^{98}(\ell n\,(k+1) - \ell n\, k)$$

(Difference series)

$\therefore$ $I = (99\ \ell n\ 99) + (-99\ \ell n\ 100 + \ell n\ 2) + (\ell n\ 99)$

$$= \ell n\left(\frac{2\times(99)^{100}}{(100)^{99}}\right) \quad ......(1)$$

For option (B) :

Now, consider $(100)^{99} = (1+99)^{99}$

$= {}^{99}C_0 + {}^{99}C_1(99) + {}^{99}C_2(99)^2 + ... + {}^{99}C_{97}(99)^{97} + \underbrace{{}^{99}C_{98}(99)^{98}}_{(\text{value}=(99)^{99})} + \underbrace{{}^{99}C_{99}(99)^{99}}_{(\text{value}=(99)^{99})}$

$\Rightarrow (100)^{99} > 2.(99)^{99} \quad \Rightarrow \frac{2\times(99)^{99}}{(100)^{99}} < 1$

$\therefore \quad \frac{2\times(99)^{100}}{(100)^{99}} < 99$ (on multiplying by 99)

$\Rightarrow \quad I < \ell n99$

For option (C) :

Since, $\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{(x+1)^2}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)dx}{x(x+1)}$

$\Rightarrow \quad \sum_{k=1}^{98}\left(\frac{1}{k+2}\right) < I$

(on integration)

$\Rightarrow \quad \underbrace{\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+......+\frac{1}{100}\right)}_{98\text{ terms}} < I$

$\Rightarrow \quad \frac{98}{100} < \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+......+\frac{1}{100} < I$

$\therefore \quad I > \frac{49}{50}$

Hence option (C) is correct.

21. Ans. (BONUS)

Sol. $g(x) = \int_{\sin x}^{\sin 2x}\sin^{-1} t\, dt$

$\Rightarrow$ $g'(x) = 2\sin^{-1}(\sin 2x) \times \cos 2x - \sin^{-1}(\sin x)\cos x$

$\Rightarrow$ $g'\left(\frac{\pi}{2}\right) = 0$ & $g'\left(-\frac{\pi}{2}\right) = 0$

No option matches the result

22. Ans. (A)

Sol. Let $I = \int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{1+e^x}dx$

$$= \int_0^{\pi/2}\left(\frac{1}{1+e^x}+\frac{1}{1+e^{-x}}\right)x^2\cos x\, dx$$

$$= \int_0^{\pi/2}\underset{(I)}{x^2}\ \underset{(II)}{\cos x}\, dx = \left(x^2\sin x\right)_0^{\pi/2} - 2\int_0^{\pi/2}\underset{(I)}{x}.\underset{(II)}{\sin x}\, dx$$

$$= \frac{\pi^2}{4} - 2\left[-\left(x.\cos x\right)_0^{\pi/2} + \int_0^{\pi/2}1.\cos x\right]$$

$$= \frac{\pi^2}{4} - 2[0+1] = \left(\frac{\pi^2}{4} - 2\right)$$

23. Ans. (B, C)

Sol.

$$\ell n f(x) = \lim_{n\to\infty}\frac{x}{n}\ell n\left[\frac{\prod_{r=1}^{n}\left(x+\frac{1}{r/n}\right)}{\prod_{r=1}^{n}\left(x^2+\frac{1}{(r/n)^2}\right)}\frac{1}{\prod_{r=1}^{n}(r/n)}\right]$$

$$= x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(\frac{x\frac{r}{n}+1}{\left(x\frac{r}{n}\right)^2+1}\right)$$

$$= x\int_0^1 \ell n\left(\frac{1+tx}{1+t^2x^2}\right)dt \qquad \text{(put } tx = z)$$

$$\ell n f(x) = \int_0^x \ell n\left(\frac{1+z}{1+z^2}\right)dz$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \ell n\left(\frac{1+x}{1+x^2}\right)$$

sign scheme of $f'(x)$: $+$ before 1, $-$ after 1

also $f'(1) = 0$

$$\Rightarrow \quad f\left(\frac{1}{2}\right) < f(1),\ f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right),\ f'(2) < 0$$

$$\text{Also } \frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ell n\left(\frac{4}{10}\right) - \ell n\left(\frac{3}{5}\right)$$

$$= \ell n\left(\frac{4}{6}\right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

24. Ans. (1)

Sol. Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$

$$\Rightarrow \quad f'(x) = \frac{x^2}{1+x^4} - 2$$

$$\text{as } \frac{1+x^4}{x^2} \ge 2 \qquad \Rightarrow \qquad \frac{x^2}{1+x^4} \le \frac{1}{2}$$

$\Rightarrow f'(x) \le -\frac{3}{2} \Rightarrow f(x)$ is continuous and decreasing

$$f(0) = 1 \text{ and } f(1) = \int_0^1 \frac{t^2}{1+t^4}dt - 2 \le -\frac{3}{2}$$

by IVT $f(x) = 0$ possesses exactly one solution in [0, 1]

25. Ans. (0)

Sol. Given $f(x) = \begin{cases} [x] & x \le 2 \\ 0 & x > 2 \end{cases}$

where [x] denotes greatest integer function.

$$\text{Now } I = \int_{-1}^{2}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$I = \int_{-1}^{0}\frac{x f(x^2)}{2+f(x+1)}dx + \int_{0}^{1}\frac{x f(x^2)}{2+f(x+1)}dx + \int_{1}^{\sqrt{2}}\frac{x f(x^2)}{2+f(x+1)}dx + \int_{\sqrt{2}}^{\sqrt{3}}\frac{x f(x^2)}{2+f(x+1)}dx + \int_{\sqrt{3}}^{2}\frac{x f(x^2)}{2+f(x+1)}dx$$

$\therefore \quad I = I_1 + I_2 + I_3 + I_4 + I_5$

Clearly I_1, I_2, I_4 & I_5 are zero using definition of $f(x)$

$$\therefore \quad I = I_3 = \int_1^{\sqrt{2}}\frac{x f(x^2)}{2+f(x+1)}dx$$

$$= \int_1^{\sqrt{2}}\frac{x.1}{2+0}dx = \left.\frac{x^2}{4}\right|_1^{\sqrt{2}} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$

$\therefore \quad 4I - 1 = 0$

26. Ans. (9)

Sol. $\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(9+\frac{3}{1+x^2}\right)dx$

$$\alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$$

$$= e^{9+\frac{3\pi}{4}} - 1$$

$$\log|\alpha+1| = 9 + \frac{3\pi}{4} \Rightarrow \text{Ans.} = 9$$

27. Ans. (7)

Sol. $\lim_{x\to1}\frac{F(x)}{G(x)} = \lim_{x\to1}\frac{F'(x)}{G'(x)}$

$$= \lim_{x\to1}\frac{f(x)}{x\left|f(f(x))\right|} = \frac{1}{14}$$

$$\Rightarrow \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \Rightarrow \left|f\left(\frac{1}{2}\right)\right| = 7$$

$$f\left(\frac{1}{2}\right) = 7$$

$f\left(\frac{1}{2}\right) \ne -7$ as $f(x)$ vanishes exactly at one point.

28. Ans. (A, C)

Sol. Let $I_1 = \int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt$

$$= \int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{\pi}^{2\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt$$

$$+ \int_{2\pi}^{3\pi} e^t\left(\sin^6 at + \cos^6 at\right)dt + \int_{3\pi}^{4\pi} e^t\left(\sin^6 at + \cos^6 at\right)dt$$

$$\therefore\ I_1 = \int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt +$$

$$\int_0^{\pi} e^{\pi+t}\left(\sin^6 at + \cos^4 at\right)dt$$

$$+ \int_0^{\pi} e^{2\pi+t}\left(\sin^6 at + \cos^4 at\right)dt +$$

$$\int_0^{\pi} e^{t+3\pi}\left(\sin^6 at + \cos^4 at\right)dt$$

$$= \left(1 + e^{\pi} + e^{2\pi} + e^{3\pi}\right)\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt$$

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt} = 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

29. Ans. (A, B)

Sol. Given $f(x) = \left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x$

$$\therefore \int_0^{\pi/4} \underbrace{x}_{I}\underbrace{\left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x}_{II}\,dx$$

Using I.B.P.

$$= x.\left(\tan^7 x - \tan^3 x\right)_0^{\pi/4} - \int_0^{\pi/4}\left(\tan^7 x - \tan^3 x\right)dx$$

$$= -\int_0^{\pi/4} \tan^3 x\left(\tan^2 x - 1\right)\sec^2 x\,dx$$

Put tanx = t

$$= \int_0^1 \left(t^3 - t^5\right)dt = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

Also,

$$\int_0^{\pi/4}\left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x\,dx$$

$$= \tan^7 x - \tan^3 x\Big|_0^{\pi/4} = 0$$

30. Ans. (D)

Sol. $f'(x) = \dfrac{192x^3}{2 + \sin^4 \pi x}$

$$\frac{192x^3}{3} \le f'(x) \le \frac{192x^3}{2}$$

$$\frac{192}{12}\left(x^4 - \frac{1}{16}\right) \le \int_{1/2}^{x} f'(x)dx \le \frac{192}{8}\left(x^4 - \frac{1}{16}\right)$$

$$16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$$

$$2.6 \le \int_{1/2}^{1} f(x)\,dx \le 3.9$$

out of given options only option (D) is correct.

31. Ans. (A, B, C)

Sol. According to given data

$F(x) < 0\ \forall\ x \in (1,3)$

$f(x) = x\,F(x)$

$f'(x) = F(x) + x\,F'(x)$ ...(i)

$f'(1) = F(1) + F'(1) < 0$

(use $F(1) = 0$ & $F'(x) < 0$)

$f(2) = 2\,F(2) < 0$

(use $F(x) < 0\ \forall\ x \in (1,3)$)

$f'(x) = F(x) + x\,F'(x) < 0$

(use $F(x) < 0\ \forall\ x \in (1,3)$)

$F'(x) < 0$

32. Ans. (C, D)

Sol. Given

$$\int_1^3 x^2 F'(x)\,dx = -12$$

$$\Rightarrow [x^2F(x)]_1^3 - 2\int_1^3 x\,F(x)\,dx = -12$$

$$\Rightarrow \int_1^3 f(x)\,dx = -12 \quad \text{Use } x\,F(x) = f(x)$$

Given

$$\int_1^3 x^3\,F''(x)\,dx = 40$$

$$\Rightarrow [x^3F'(x)]_1^3 - 3\int_1^3 x^2F'(x)\,dx = 40$$

$$\Rightarrow \left[x^2(f'(x) - F(x)\right]_1^3 = 4$$

$$9(f'(3) - F(3)) - \left(f'(1) - F(1)\right) = 4$$

$$9f'(3) + 36 - f'(1) = 4$$

$$9f'(3) - f'(1) + 32 = 0$$

33. Ans. (A, C)

Sol. Given that $f : [a, b] \to [1, \infty)$

$$g(x) = \begin{cases} 0 & x < a \\ \int_a^x f(t)dt, & a \le x \le b \\ \int_a^b f(t)dt, & x > b \end{cases}$$

Now $g(a^-) = 0 = g(a^+) = g(a)$

$g(b^-) = g(b^+) = g(b) = \int_a^b f(t)dt$

$\Rightarrow$ g is continuous $\forall$ x $\in$ R

$$\text{Now } g'(x) = \begin{cases} 0 & : & x < a \\ f(x) & : & a < x < b \\ 0 & : & x > b \end{cases}$$

$g'(a^-) = 0$ but $g'(a^+) = f(a) \ge 1$

$\Rightarrow$ g is non differentiable at x = a

and $g'(b^+) = 0$ but $g'(b^-) = f(b) \ge 1$

$\Rightarrow$ g is non differentiable at x = b

34. Ans. (A, C, D)

Sol. $f(x) = \int_{\frac{1}{x}}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$f'(x) = 1.\frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \left(\frac{-1}{x^2}\right)\frac{e^{-\left(\frac{1}{x}+x\right)}}{1/x}$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

$\therefore$ $f(x)$ is monotonically increasing on $(0, \infty)$

$\Rightarrow$ A is correct & B is wrong.

Now $f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^{x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt + \int_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$= 0 \ \forall \ x \in (0, \infty)$

Now let $g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$g(-x) = f(2^{-x}) \qquad = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = -g(x)$

$\therefore$ $f(2^x)$ is an odd function.

35. Ans. (2)

Sol. using integration by part

$$\int_0^1 4x^3 \left(\left(1-x^2\right)^5\right)'' dx$$

$$= 4x^3\left(\left(1-x^2\right)^5\right)'\Big]_0^1 - \int_0^1 12x^2\left(\left(1-x^2\right)^5\right)' dx$$

using integration by part

$$= -12\left[x^2\left(\left(1-x^2\right)^5\right)\Big|_0^1 - \int_0^1 2x\left(1-x^2\right)^5 dx\right]$$

$$= 12.2.\int_0^1 x(1-x^2)^5 dx$$

Let $1 - x^2 = t \Rightarrow \quad xdx = -\frac{dt}{2}$

$$= 24\int_1^0 t^5\left(-\frac{dt}{2}\right)$$

$$= 12\int_0^1 t^5 dt = 2$$

36. Ans. (A)

Sol. Let $\text{cosec}x + \cot x = e^u$

$\text{cosec}x - \cot x = e^{-u}$

$\text{cosec}x = \frac{1}{2}(e^u + e^{-u})$ & $\cot x = \frac{1}{2}(e^u - e^{-u})$

$\text{cosec}^2 x dx = -\frac{1}{2}(e^u + e^{-u})du$

$$\Rightarrow \int_{\ell n(\sqrt{2}+1)}^{0} 2^{17}\left(\frac{1}{2}(e^u + e^{-u})\right)^{15}\left\{-\frac{1}{2}(e^u + e^{-u})\right\}du$$

$$= \int_0^{\ell n(\sqrt{2}+1)} 2(e^u + e^{-u})^{16} du.$$

37. Ans. (B)

Sol. $f(x) = \int_0^{x^2} f(\sqrt{t})dt$

$f'(x) = 2x f(x) \qquad \because x \in [0, 2]$

$\Rightarrow f'(x) = 2x f(x) \qquad \because F'(x) = f'(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = 2x \Rightarrow \ell n(f(x)) = x^2 + c$

$c = 0$ $(\because f(0) = 1)$

$\Rightarrow f(x) = e^{x^2}$

$f(x) = \int_0^{x^2} f(\sqrt{t})dt = \int_0^{x^2} e^t . dt$

$\therefore f(2) = \int_0^{x^2} e^t\, dt = e^4 - 1$.

38. Ans. (A)

Sol.

$g\left(\frac{1}{2}\right) = \lim_{h\to 0^+} \int_h^{1-h} \frac{dt}{\sqrt{t(1-t)}}$

$= \lim_{h\to 0^+} \int_h^{1-h} \frac{dt}{\sqrt{\frac{1}{4} - (t - \frac{1}{2})^2}}$

$= \sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_0^1 = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$

39. Ans. (D)

Sol. Given,

$g(a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$

$g^1(a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}(-\ln t + \ln(1-t)dt$

$g'\left(\frac{1}{2}\right) = \lim_{h\to 0^+} \int_h^{1-h} \frac{\ell n\left(\frac{1-t}{t}\right)dt}{\sqrt{t(1-t)}}$ ...(i)

$g^1\left(\frac{1}{2}\right) \lim_{h\to 0^+} \int_h^{1-h} \frac{\ln\left(\frac{1-(1-t)}{1-t}\right)}{\sqrt{(1-t)t}}dt$ ...(ii)

(Apply $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$)

$\Rightarrow 2g^1\left(\frac{1}{2}\right) = \lim_{h\to 0^+} \int_h^{1-h} 0\, dt \Rightarrow g^1\left(\frac{1}{2}\right) = 0$

40. Ans. (D)

Sol. (P) let $f(x) = ax^2 + bx + c$ where a,b,c are non negative integers

$f(0) = c = 0$ ...(1)

and $\int_0^1 \left(ax^2 + bx\right)dx = 1$

$= \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = \frac{a}{3} + \frac{b}{2} = 1$

$\Rightarrow 2a + 3b = 6$

$\Rightarrow$ a = 3 & b = 0 OR a = 0 & b = 2

(Q) maximum of $\sin x^2 + \cos x^2 = \sqrt{2}$

$\Rightarrow \sin\left(\frac{\pi}{4} + x^2\right) = 1$ but $x^2 \in [0,13]$

$\Rightarrow \frac{\pi}{4} + x^2 = (4n+1)\frac{\pi}{2}$

$\Rightarrow$ which is satisfied for n = 0 & 1

$\Rightarrow$ 4 solutions

(R) $I = \int_{-2}^{2} \frac{3x^2}{1+e^x}dx$ put x

$= -t$ $I = -\int_2^{-2} \frac{3t^2e^t dt}{1+e^t}$

$\Rightarrow 2I = \int_{-2}^{2} \frac{3x^2\left(1+e^x\right)}{1+e^x}dx = 2\int_0^2 3x^2dx$

$\Rightarrow I = \left[x^3\right]_0^2 = 8$

(S) $\dfrac{\int_{-\frac{1}{2}}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx}{\int_0^{1/2} \cos 2x \log\frac{(1+x)}{(1-x)}dx}$

$= \dfrac{\int_{-1/2}^{1/2} (\text{odd function})dx}{\int_0^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx} = 0$