DIFFERENTIABILITY

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \to \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)\,g(x)$. Then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) If g is continuous at x = 1, then fg is differentiable at x = 1

(B) If fg is differentiable at x = 1, then g is continuous at x = 1

(C) If g is differentiable at x = 1, then fg is differentiable at x = 1

(D) If fg is differentiable at x = 1, then g is differentiable at x = 1

2. Let the functions $f : (-1, 1) \to \mathbb{R}$ and $g : (-1,1) \to (-1, 1)$ be defined by

$f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,

where [x] denotes the greatest integer less than or equal to x. Let $fog:(-1, 1) \to \mathbb{R}$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval (–1, 1) at which fog is **NOT** continuous, and suppose d is the number of points in the interval (–1, 1) at which fog is **NOT** differentiable. Then the value of c + d is _____. **[JEE(Advanced) 2020]**

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in \mathbb{R}$. If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE?

[JEE(Advanced) 2020]

(A) f is differentiable at every $x \in \mathbb{R}$

(B) If g(0) = 1, then g is differentiable at every $x \in \mathbb{R}$

(C) The derivative $f'(1)$ is equal to 1

(D) The derivative $f'(0)$ is equal to 1

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced) 2018]**

5. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, [t] denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

[JEE(Advanced) 2018]

List-I	List-II
P. The function f_1 is	**1.** **NOT** continuous at x = 0
Q. The function f_2 is	**2.** continuous at x = 0 and **NOT** differentiable at x = 0
R. The function f_3 is	**3.** differentiable at x = 0 and its derivative is **NOT** continuous at x = 0
S. The function f_4 is	**4.** differentiable at x = 0 and its derivative is continuous at x = 0

The correct option is :

(A) P $\to$ 2; Q $\to$ 3, R $\to$ 1; S $\to$ 4

(B) P $\to$ 4; Q $\to$ 1; R $\to$ 2; S $\to$ 3

(C) P $\to$ 4; Q $\to$ 2, R $\to$ 1; S $\to$ 3

(D) P $\to$ 2; Q $\to$ 1; R $\to$ 4; S $\to$ 3

6. Let a, b $\in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is -

[JEE(Advanced) 2016]

(A) differentiable at x = 0 if a = 0 and b = 1

(B) differentiable at x = 1 if a = 1 and b = 0

(C) **NOT** differentiable at x = 0 if a = 1 and b = 0

(D) **NOT** differentiable at x = 1 if a = 1 and b = 1

7. Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be function defined by $f(x)=[x^2-3]$ and $g(x) = |x|\,f(x) + |4x - 7|\,f(x)$, where [y] denotes the greatest integer less than or equal to y for y $\in \mathbb{R}$. Then

[JEE(Advanced) 2016]

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2},2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2},2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2},2\right)$

8. Let g : $\mathbb{R} \to \mathbb{R}$ be a differentiable functions with g(0) = 0, g'(0) = 0 and g'(1) $\neq$ 0. Let

$$f(x)=\begin{cases}\frac{x}{|x|}g(x) & , \quad x\neq 0\\ 0 & , \quad x=0\end{cases}$$

and $h(x) = e^{|x|}$ for all x $\in \mathbb{R}$. Let $(f\circ h)(x)$ denote $f(h(x))$ and $(h\circ f)(x)$ denote $h(f(x))$. Then which of the following is(are) true ?

[JEE(Advanced) 2015]

(A) f is differentiable at x = 0

(B) h is differentiable at x = 0

(C) $f$oh is differentiable at x = 0

(D) hof is differentiable at x = 0

9. Let $f: \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : \mathbb{R} \to \mathbb{R}$ by **[JEE(Advanced) 2014]**

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is

10. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : [0, \infty) \to \mathbb{R}$, $f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \to [0, \infty)$ be defined by **[JEE(Advanced) 2014]**

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \ge 0; \end{cases}$$

$f_2(x) = x^2$;

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \ge 0 \end{cases}$$

and

$$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \ge 0. \end{cases}$$

	List-I		**List-II**
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	$f_2 o f_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

Codes :

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

SOLUTIONS

1. Ans. (A,C)

Sol. $f: R \to R \quad f(x) = (x^2 + \sin x)(x-1)\ f(1^+)$

$= f(1^-) = f(1) = 0$

$fg(x) : f(x).g(x) \quad fg : R \to R$

let $fg(x) = h(x) = f(x).g(x) \qquad h : R \to R$

option (c) $\quad h'(x) = f'(x)g(x) + f(x)\,g'(x)$

$h'(1) = f'(1)\,g(1) + 0,$

(as $f(1) = 0$, $g'(x)$ exists}

$\Rightarrow$ if g(x) is differentiable then h(x) is also differentiable (true)

option (A) If g(x) is continuous at x = 1 then $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h\to0^+}\frac{h(1+h)-h(1)}{h}$$

$$h'(1^+) = \lim_{h\to0^+}\frac{f(1+h)g(1+h)-0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h\to0^+}\frac{f(1-h)g(1-h)-0}{-h} = f'(1)g(1)$$

So $\quad h(x) = f(x).g(x)$ is differentiable

at x = 1 $\qquad$ (True)

option (B) $\quad$ (D)

$$h'(1^+) = \lim_{h\to0^+}\frac{h(1+h)-h(1)}{-h}$$

$$h'(1^+) = \lim_{h\to0^+}\frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h\to0^+}\frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$\Rightarrow g(1^+) = g(1^-)$

So we cannot comment on the continuity and differentiability of the function.

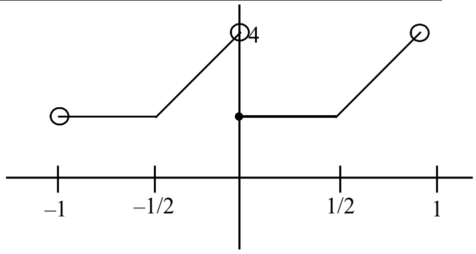
2. Ans. (4)

Sol. $f(x) = |2x-1| + |2x+1|$

$g(x) = \{x\}$

$f(g(x)) = |2\{x\}-1| + |2\{x\}+1|$

$$= \begin{cases} 2 & \{x\} \le 1/2 \\ 4\{x\} & \{x\} > 1/2 \end{cases}$$



discontinuous at $x = 0 \Rightarrow c = 1$

Non differential at $x = -\tfrac{1}{2},\ 0,\ \tfrac{1}{2} \Rightarrow d = 3$

$\therefore \quad c + d = 4$

3. Ans. (A, B, D)

Sol. since $f(x) = xg(x)$

$$\lim_{x\to0} f(x) = \lim_{x\to0} xg(x)$$

$$\lim_{x\to0} f(x) = \left(\lim_{x\to0} x\right).\left(\lim_{x\to0} g(x)\right)$$

$$\lim_{x\to0} f(x) = 0\times1 = 0 \qquad ...(1)$$

$f(x+y) = f(x) + f(y) + f(x)\,f(y)$

Now we check continuity of $f(x)$

at x = a

$$\lim_{h\to0} f(a+h) = f(a) + f(h) + f(a)f(h)$$

$$\lim_{h\to0}\left(f(a) + f(h)(1+f(a))\right)$$

$$\lim_{h\to0} f(a+h) = f(a)$$

$\therefore f(x)$ is continuous $\forall\ x \in R$

$$\lim_{x\to0} f(x) = f(0) = 0 \quad \left(\lim_{x\to0} f(x) = 0\right)$$

$\therefore f(0) = 0$

and $\displaystyle\lim_{x\to0}\frac{f'(x)}{1} = 1$

$\therefore f'(0) = 1$

Now

$f(x+y) = f(x) + f(y) + f(x)\,f(y)$

using partial derivative (w.r.t. y)

$f'(x+y) = f'(y) + f(x)\,f'(y)$

put y = 0

$f'(x) = f'(0) + f(x)\,f'(0)$

$f'(x) = 1 + f(x)$

$$\int\frac{f'(x)}{1+f(x)}dx = \int 1\,dx$$

$\ell n\left|\left(1+f(x)\right)\right| = x + C$

$f(0) = 0$; $c = 0$ $\quad \therefore \left|1+f(x)\right| = e^x$

$1 + f(x) = \pm e^x$ or $f(x) = \pm e^x - 1$

Now $f(0) = 0 \therefore f(x) = e^x - 1$

$\therefore f(x) = e^x - 1$

option (A) is correct

and $f'(x) = e^x$

$f'(0) = 1$ option(D) is correct

$$g(x) = \frac{f(x)}{x} = \begin{Bmatrix} \frac{e^x - 1}{x} & ; & x \neq 0 \\ 1 & ; & x = 0 \end{Bmatrix}$$

$$g'(0+h) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{e^h - 1}{h} - 1}{h} = \frac{1}{2}$$

option (B) is correct

4. Ans. (2)

Sol. $P(x, y) : f(x + y) = f(x)f'(y) + f'(x) f(y) \ \forall \ x, y \in R$

$P(0, 0) : f(0) = f(0)f'(0) + f'(0) f(0)$

$\Rightarrow 1 = 2f'(0)$

$\Rightarrow f'(0) = \frac{1}{2}$

$P(x, 0) : f(x) = f(x). f'(0) + f'(x).f(0)$

$\Rightarrow f(x) = \frac{1}{2} f(x) + f'(x)$

$\Rightarrow f'(x) = \frac{1}{2} f(x)$

$\Rightarrow f(x) = e^{\frac{1}{2}x}$

$\Rightarrow \ln(f(4)) = 2$

5. Ans. (D)

Sol. (i) $f(x) = \sin\sqrt{1 - e^{-x^2}}$

$$f_1'(x) = \cos\sqrt{1-e^{-x^2}} . \frac{1}{2\sqrt{1-e^{-x^2}}}\left(0 - e^{-x^2}.(-2x)\right)$$

$f_1'(x)$ does not exist at $x = 0$

So. $P \to 2$

(ii) $$f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x\to 0^+} \frac{\sin x}{x} \frac{x}{\tan^{-1} x} = 1$$

$\Rightarrow f_2(x)$ does not continuous at $x = 0$

So $Q \to 1$

(iii) $f_3(x) = \left[\sin \ell n(x+2)\right] = 0$

$1 < x + 2 < e^{\pi/2}$

$\Rightarrow 0 < \ell n(x + 2) < \frac{\pi}{2}$

$\Rightarrow 0 < \sin(\ell n(x + 2)) < 1$

$\Rightarrow f_3(x) = 0$

So $R \to 4$

(iv) $$f_4(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

So $S \to 3$

6. Ans. (A, B)

Sol. If $x^3 - x \geq 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

$x^3 - x < 0 \Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$

Similarly

$b|x|\sin|x^3 + x| = bx\sin(x^3 + x)$ for all $x \in R$

$\therefore \quad f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x)$

which is composition and sum of differentiable functions

therefore always continuous and differentiable.

7. Ans. (B, C)

Sol. $f(x) = [x^2] - 3$

$g(x) = (|x| + |4x - 7|)([x^2] - 3)$

$\because \quad f$ is discontinuous at in $\left[-\frac{1}{2}, 2\right]$

and $|x| + |4x - 7| \neq 0$ at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$\Rightarrow \quad g(x)$ is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}$ in $\left(-\frac{1}{2}, 2\right)$

In $(0 - \delta, 0 + \delta)$

$g(x) = (|x| + |4x - 7|).(-3)$

$\Rightarrow$ 'g' is non derivable at $x = 0$.

In $\left(\frac{7}{4}-\delta, \frac{7}{4}+\delta\right)$

$g(x) = 0$ as $f(x) = 0$

$\Rightarrow$ Derivable at $x = \frac{7}{4}$

$\therefore$ 'g' is non-derivable at $0, 1, \sqrt{2}, \frac{7}{4}$

8. Ans. (A, D)

Sol. (A) $f(x) = \begin{cases} g(x) & , x > 0 \\ 0 & , x = 0 \\ -g(x) & , x < 0 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} g'(x) & , x \geq 0 \\ -g'(x) & , x < 0 \end{cases}$

so 'A' is right

(B) $h(x) = \begin{cases} e^{x} & , x \geq 0 \\ e^{-x} & , x < 0 \end{cases}$

$\Rightarrow h'(x) = \begin{cases} e^{x} & , x > 0 \\ -e^{-x} & , x < 0 \end{cases}$

$h'(0^+) = 1, h'(0^-) = -1, \therefore$ B is wrong

(C) $f(h(x)) = g(h(x))$ as $h(x) > 0$

$z = g\left(e^{|x|}\right) = \begin{cases} g\left(e^{x}\right) & , x \geq 0 \\ g\left(e^{-x}\right) & , x < 0 \end{cases}$

$z' = \begin{cases} g'\left(e^{x}\right)e^{x} & , x > 0 \\ -g'\left(e^{-x}\right)e^{-x} & , x < 0 \end{cases}$

$z'(0^+) = g'(1), z'(0^-) = -g'(1)$ and

$g'(1) \# -g'(1)$

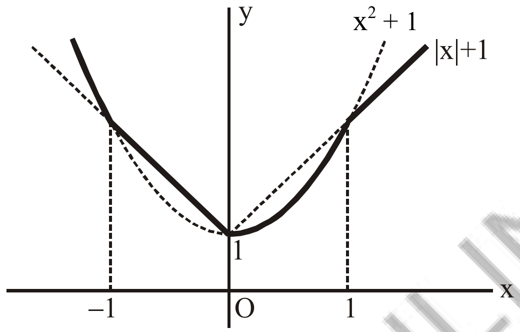
so C is wrong

(D) $\lim\limits_{x \to 0} \frac{e^{|g(x)|} - 1}{|g(x)|} \cdot \frac{|g(x)|}{x}$

$\lim\limits_{x \to 0} \frac{e^{|g(x)|} - 1}{|g(x)|} \cdot \frac{|g(x) - 0|}{x} \cdot \frac{|X|}{X} = 0$

9. Ans. (3)

Sol.

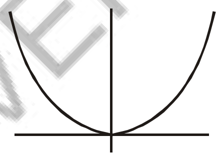


h(x) is not differentiable at $x = \pm 1$ & 0

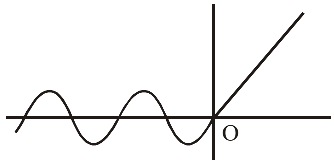
10. Ans. (D)

Sol. (P) $f_4(x) = \begin{cases} |x|^2 & ; x < 0 \\ e^{2x} - 1 & ; x \geq 0 \end{cases}$

$f_4(x)$ is onto and one-one



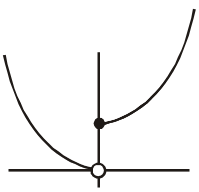
(Q)



RHD = LHD = 1, $f_3(x)$ is differentiable

But not one-one

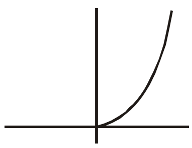
(R)



$f_2(f_1(x)) = \begin{cases} x^2 & ; x < 0 \\ e^{2x} & ; x \geq 0 \end{cases}$

Neither continuous nor one-one

(S)



Continuous and one-one function