FUNCTION

1. Let $|M|$ denote the determinant of a square matrix M. Let g : $\left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be the function defined by

$$g(\theta) = \sqrt{f(\theta)-1} + \sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$$

where

$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}.$$

Let p(x) be a quadratic polynomial whose roots are the maximum and minimum values of the function g(θ), and p(2) = $2 - \sqrt{2}$. Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$

(B) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$

(C) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$

(D) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

2. If the function $f : R \to R$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE** ? **[JEE(Advanced) 2020]**

(A) f is one-one, but **NOT** onto

(B) f is onto, but **NOT** one-one

(C) f is **BOTH** one-one and onto

(D) f is **NEITHER** one-one **NOR** onto

3. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \frac{4^x}{4^x + 2}$

Then the value of $f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \dots\dots\dots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$ is____. **[JEE(Advanced) 2020]**

4. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is _____ . **[JEE(Advanced) 2018]**

5. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$ Then :- **[JEE(Advanced)-2014]**

(A) $f(x)$ is an odd function

(B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function

(D) $f(x)$ is an even function

SOLUTIONS

1. Ans. (A, C)

Sol. $f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+$

$$\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\frac{\pi}{4} & \tan\pi\end{vmatrix}$$

$f(\theta)=\frac{1}{2}\begin{vmatrix}2 & \sin\theta & 1\\ 0 & 1 & \sin\theta\\ 0 & -\sin\theta & 1\end{vmatrix}+$

$$\begin{vmatrix}0 & -\sin\left(\theta-\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & 0 & \log_e\left(\frac{4}{\pi}\right)\\ -\tan\left(\theta-\frac{\pi}{4}\right) & -\log_e\left(\frac{4}{\pi}\right) & 0\end{vmatrix}$$

$f(\theta) = (1 + \sin^2\theta) + 0$ (skew symmetric)

$g(\theta)=\sqrt{f(\theta)-1}+\sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$

$= |\sin\theta| + |\cos\theta| \quad$ for $\theta\in\left[0,\frac{\pi}{2}\right]$

$g(\theta)\in\left[1,\sqrt{2}\right]$

Again let $P(x)=k\left(x-\sqrt{2}\right)(x-1)$

$2-\sqrt{2}=k\left(2-\sqrt{2}\right)(2-1)$

$\Rightarrow k=1 \qquad (P(2)=2-\sqrt{2}$ given)

$\therefore\ P(x)=\left(x-\sqrt{2}\right)(x-1)$

for option (A) $P\left(\frac{3+\sqrt{2}}{4}\right)<0$ correct

option (B) $P\left(\frac{1+3\sqrt{2}}{4}\right)<0$ incorrect

option (C) $P\left(\frac{5\sqrt{2}-1}{4}\right)>0$ correct

option (D) $P\left(\frac{5-\sqrt{2}}{4}\right)>0$ incorrect

2. Ans. (C)

Sol. $f(x)$ is a non-periodic, continuous and odd function

$$f(x)=\begin{cases}-x^2+x\sin x, & x<0\\ x^2-x\sin x, & x\ge 0\end{cases}$$

$f(-\infty)=\lim_{x\to-\infty}(-x^2)\left(1-\frac{\sin x}{x}\right)=-\infty$

$f(\infty)=\lim_{x\to\infty}x^2\left(1-\frac{\sin x}{x}\right)=\infty$

$\Rightarrow$ Range of $f(x)$ = R

$\Rightarrow$ $f(x)$ is an onto function ...(1)

$$f'(x)=\begin{cases}-2x+\sin x+x\cos x, & x<0\\ 2x-\sin x-x\cos x, & x\ge 0\end{cases}$$

For $(0, \infty)$

$f'(x) = (x - \sin x) + x(1 - \cos x)$

always +ve or 0   always +ve or 0

$\Rightarrow$ $f'(x) > 0$

$\Rightarrow$ $f'(x)\ge 0,\ \forall\, x\in(-\infty,\infty)$

equality at $x = 0$

$\Rightarrow$ $f(x)$ is one-one function ... (2)

From (1) & (2), $f(x)$ is both one-one & onto.

3. Ans. (19.00)

Sol. $f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}$

$=\frac{4^x}{4^x+2}+\frac{4/4^x}{\frac{4}{4^x}+2}$

$=\frac{4^x}{4^x+2}+\frac{4}{4+2.4^x}$

$=\frac{4^x}{4^x+2}+\frac{2}{2+4^x}=1$

so, $f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+\ldots+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$

$=19+f\left(\frac{1}{2}\right)-f\left(\frac{1}{2}\right)=19$

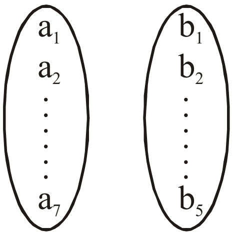
4. Ans. (119)

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \rightarrow$ Number of one-one function = ${}^7C_5 \times 5!$

$\beta \rightarrow$ Number of onto function Y to X



1, 1, 1, 1, 3   1, 1, 1, 2, 2

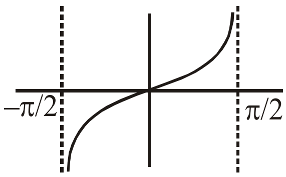
$$\frac{7!}{3!4!}\times 5! + \frac{7!}{(2!)^3 3!}\times 5!$$

$$= \left({}^7C_3 + 3.{}^7C_3\right)5! = 4\times{}^7C_3 \times 5!$$

$$\frac{\beta-\alpha}{5!} = 4\times{}^7C_3 - {}^7C_5 = 4\times 35 - 21 = 119$$

5. Ans. (A, B, C)

Sol. $f(x) = \left(\ell n\left(\sec x + \tan x\right)\right)^3$



$$f'(x) = \frac{3\left(\ell n\left(\sec x + \tan x\right)\right)^2\left(\sec x \tan x + \sec^2 x\right)}{\left(\sec x + \tan x\right)} > 0$$

$f(x)$ is an increasing function

$$\lim_{x\to -\frac{\pi}{2}} f(x) \rightarrow -\infty \ \ \& \ \lim_{x\to \frac{\pi}{2}} f(x) \rightarrow \infty$$

Range of $f(x)$ is R and onto function

$$f(-x) = \left(\ell n\left(\sec x - \tan x\right)\right)^3 = \left(\ell n\left(\frac{1}{\sec x + \tan x}\right)\right)^3$$

$f(-x) = -(\ell n(\sec x + \tan x))^3$

$f(x) + f(-x) = 0 \Rightarrow f(x)$ is an odd function.