FORMULAE OF MATHEMATICS

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CONTENTS

Ratio and Proportion   Chapter 1

ELEMENTARY LAWS AND RESULTS ON RATION AND PROPORTION

1. $\frac{a}{b} > 1$ Implies $\frac{\mathrm{a}}{\mathrm{b}} > \frac{a+x}{b+x}$ $(x > 0)$
2. $\frac{a}{b} < 1$ implies $\frac{a}{b} < \frac{a+x}{b+x}$ $(x > 0)$
3. If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$ are unequal fractions, of which the denominators are of the same sign, then the fraction $\frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n}$ lies, in magnitude, between the greatest and least of them.
4. If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_k}{b_k}$, then each of these ratios is equal to $\left(\frac{\alpha_1 a_1^n + \alpha_2 a_2^n + \ldots + \alpha_k a_k^n}{\alpha_1 b_1^n + \alpha_2 b_2^n + \ldots + \alpha_k b_k^n}\right)^{1/n}$, where n is an integer.
5. a, b, c, d are in proportion. Then

(i) $\frac{a}{b} = \frac{c}{d}$

(ii) $\frac{a}{c} = \frac{b}{d}$ (alternendo)

(iii) $\frac{a+b}{b} = \frac{c+d}{d}$ (componendo)

(iv) $\frac{a-b}{b} = \frac{c-d}{d}$ (dividendo)

(v) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (componendo and dividendo)

(vi) $\frac{b}{a} = \frac{d}{c}$ (invertendo)

Set Theory

Chapter 2

SET

A well defined collection of distinct objects is called a *set*. When we say, 'well defined', we mean that there must be given a rule or rules with the help of which we should readily be able to say that whether a particular object is a member of the set or is not a member of the set. The sets are generally denoted by capital letters A,B,C,….X,Y,Z.

The members of a set are called its **elements**. The elements of a set are denoted by small letters *a*, *b*, *c*,.….,*x, y, z*.

If an element *a* belong to a set A than we write $a \in A$ and if *a* does not belong to set A then we write $a \notin A$.

LAW OF ALGEBRA OF SETS

1. **Idempotent Laws :** *For any set A,*
 (i) $A \cup A = A$ (ii) $A \cap A = A$.
2. **Identity Laws :** *For any set A,*
 (i) $A \cup \phi = A$ (ii) $A \cap U = A$
 i.e., ϕ and U are identity elements for union and intersection respectively.
3. **Commutative Laws :** *For any two sets A and B,*
 (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
 i.e. union and intersection are commutative.
4. **Associative Laws :** *If A, B and C are any three sets, then*
 (i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$
 i.e. union and intersection are associative.
5. **Distributive Laws :** *If A, B and C are any three sets, then*
 (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 i.e. union and intersection are distributive over intersection and union respectively.
6. **De-Morgan's Laws :** *If A and B are any two sets, then*
 (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
7. **More results on operations on sets**
 If A and B are any two sets, then
 (i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$
 (iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$
 (v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$
 (vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.
 If A, B and C are any three sets, then
 (i) $A - (B \cap C) = (A - B) \cup (A - C)$ (ii) $A - (B \cup C) = (A - B) \cap (A - C)$
 (iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (iv) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
8. **Important results on number of elements in sets**
 If A, B and C are finite sets, and U be the finite universal set, then
 (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
 (ii) $n(A \cap B) = n(A) + n(B) \Leftrightarrow A, B$ *are disjoint non-void sets.*

(iii) $n(A-B)=n(A)-n(A\cap B)$ i.e., $n(A-B)+n(A\cap B)=n(A)$

(iv) $n(A \,\Delta\, B) = n(A)+n(B)-2n(A\cap B)$

(v) $n(A\cup B\cup C)$

$$=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C)$$

(vi) *No. of elements in exactly two of the sets A, B, C*

$$=n(A\cap B)+n(B\cap C)+n(C\cap A)-3n(A\cap B\cap C)$$

(vii) *No. of elements in exactly one of the sets A, B, C*

$$=n(A)+n(B)+n(C)-2n(A\cap B)-2n(B\cap C)-2n(A\cap C)+3n(A\cap B\cap C)$$

(viii) $n(A'\cup B')=n\left((A\cup B)'\right)=n(U)-n(A\cap B)$

(ix) $n(A'\cap B')=n\left((A\cup B)'\right)=n(U)-n(A\cup B)$.

9. **Cartesian product of two or more sets**

The Cartesian product of $n(\geq 2)$ sets $A_1, A_2, \ldots\ldots, A_n$ is defined as the set of all ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where $a_i \in A_i\ (1\leq i\leq n)$. The Cartesian product of $A_1, A_2, \ldots\ldots, A_n$ is denoted by $A_1\times A_2\times\ldots\ldots\times A_n$ or by $\prod_{i=1}^{n} A_i$.

Symbolically $\prod_{i=1}^{n} A_i=\{(a_1, a_2, \ldots\ldots, a_n): a_i\in A_i, 1\leq i\leq n\}$. If at least one of $A_1, A_2, \ldots\ldots, A_n$ is empty set, then $A_1, A_2, \ldots\ldots, A_n$ is defined as empty set. If $A_1, A_2, \ldots\ldots, A_n$ are finite sets, then

$$n\,(A_1, A_2, \ldots\ldots, A_n) = n(A_1)\times n(A_2)\times\ldots\ldots\times n(A_n).$$

10. **Important results**

(i) $A\times B\neq B\times A$ (ii) $A\times\phi=\phi\times A=\phi$

(iii) If A and B are finite sets, then $n(A\times B)=n(A)\times n(B)=n(B\times A)$

(iv) If $A\subseteq B$, then $A\times C\subseteq B\times C$

(v) (a) $A\times(B\cup C)=(A\times B)\cup(A\times C)$ (b) $A\times(B\cap C)=(A\times B)\cap(A\times C)$

(c) $A\times(B-C)=(A\times B)-(A\times C)$

(vi) $A\subseteq B, C\subseteq D\Rightarrow A\times C\subseteq B\times D$

(a) $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D)$ (b) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$.

Complex Numbers Chapter 3

DEFINITIONS

A number of the form $x+iy$, where $x, y \in R$ and $i=\sqrt{-1}$ is called a complex number and is usually denoted by z.

Thus, $z = x+iy$

Here x is called **Real Part** and y the **Imaginary Part** of z, denoted by $R_e(z)$ and $I_m(z)$ respectively.

$\therefore \quad R_e(z) = x \quad$ and $\quad I_m(z) = y$

A complex number z is said to be ***purely real*** if $I(z)=0$ and is said to be ***purely imaginary*** if $R(z)=0$. Note that the complex number $0 = 0+i0$ is both purely real and purely imaginary. It is the only complex number with this property.

We denote the set of all complex numbers by C. That is, $C = \{a+ib \mid a, b \in R\}$ Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal if $a_1 = a_2$ and $b_1 = b_2$.

Integral powers of iota *(i)* : Since $i=\sqrt{-1}$ hence we have $i^2=-1,\ i^3=-i$ and $i^4=1$.

In general $\quad i^{4n}=1, \quad i^{4n+1}=i, \quad i^{4n+2}=-1, \quad i^{4n+3}=-i, \quad$ where n is any integer.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

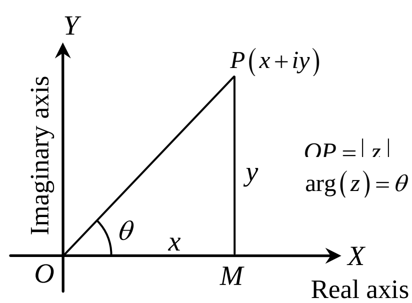
1. *Addition :* $(a+ib)+(c+id) = (a+c)+i(b+d)$
2. *Subtraction :* $(a+ib)-(c+id) = (a-c)+i(b-d)$
3. *Multiplication :* $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$
4. *Reciprocal :* $\dfrac{1}{(a+ib)}$ (where at least one of a, b is non-zero) is given by

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

5. *Division :* $(a+ib)/(c+id)$ (where at least one of c, d is non-zero)

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS



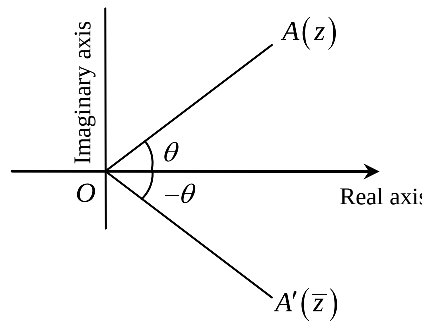
A complex number $z=x+iy$ can be represented by a point P whose Cartesian co-ordinates are (x, y) referred to rectangular axes Ox and Oy. Here the x-axis and y-axis are usually called the real and *imaginary* axes respectively. The plane is called the *Argand plane*, *complex plane* or *Gaussian plane*. The point $P(x, y)$ is called the *image* of the complex number z and z is said to be the *affix* or *complex co-ordinate* of point P.

We have $OP = \sqrt{x^2+y^2} = |z|$.

Thus, $|z|$ is the length of OP.

Note : All purely real numbers lie on the real axis and all purely imaginary numbers lie on the imaginary axis. The complex number $0 = 0+i0$ lies at the origin O.

CONJUGATE OF COMPLEX NUMBER



Let $z = a + ib$ be a complex number. We define conjugate of z, denoted by $\overline{z}$ to be the complex number $a - ib$.
That is, if $z = a + ib$, then $\overline{z} = a - ib$.

PROPERTIES OF CONJUGATE OF A COMPLEX NUMBER

If z is a complex number, then

(i) $\overline{(\overline{z})} = z$ (ii) $z + \overline{z} = 2\ Re\ (z)$

(iii) $z - \overline{z} = 2i\ \text{Im}\ (z)$

(iv) $z = \overline{z} \Leftrightarrow z$ is purely real (v) $z + \overline{z} = 0 \Leftrightarrow z$ is purely imaginary.

(vi) $z\,\overline{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2 = |\,z\,|^2$

If z_1 and z_2 are two complex numbers, then

(vii) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ (viii) $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$

(ix) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$ (x) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$ if $z_2 \neq 0$

(xi) If $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ where $a_0, a_1, ... a_n$ and z are complex number, then

$$\overline{P(z)} = \overline{a}_0 + \overline{a}_1 (\overline{z}) + \overline{a}_2 (\overline{z})^2 + ... + \overline{a}_n (\overline{z})^n = \overline{P}(\overline{z})$$

where $\overline{P}(z) = \overline{a}_0 + \overline{a}_1 z + \overline{a}_2 z^2 + ... + \overline{a}_n z^n$

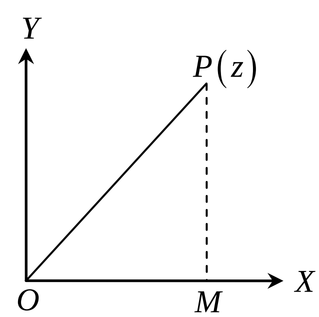
(xii) If $R(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials in z, and $Q(z) \neq 0$, then

$$\overline{R(z)} = \frac{\overline{P}(\overline{z})}{\overline{Q}(\overline{z})}, \qquad \text{e.g.} \quad \overline{\left(\frac{z + 3z^2}{z - 1}\right)} = \frac{\overline{z} + 3\overline{z}^2}{\overline{z} - 1}$$

(xiii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{vmatrix}$, then $\overline{z} = \begin{vmatrix} \overline{a}_1 & \overline{a}_2 & \overline{a}_3 \\ \overline{b}_1 & \overline{b}_2 & \overline{b}_3 \\ \overline{c}_1 & \overline{c}_2 & \overline{c}_3 \end{vmatrix}$

where $a_i, b_i, c_i\ (i = 1, 2, 3)$ are complex numbers.

MODULUS OF A COMPLEX NUMBER



Let $z = a + ib$ be a complex number. We define *modulus* of z to be the real number $\sqrt{a^2 + b^2}$ and denote it by $|\,z\,|$.

Geometrically $|\,z\,|$ represents the distance of point P from the origin, *i.e.* $|\,z\,| = OP$.

If $|\,z\,| = 1$ the corresponding complex number is known as **unimodular complex number.** Clearly z lies on a circle of unit radius having centre $(0, 0)$.

Note that $|\,z\,| \geq 0\ \forall\ z \in C$

PROPERTIES OF MODULUS

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \le \text{Re}(z) \le |z|$

(iv) $-|z| \le \text{Im}(z) \le |z|$

(v) $z\,\bar{z} = |z|^2$

If z_1, z_2 are two complex numbers, then

(vi) $|z_1 \; z_2| = |z_1|\,|z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \ne 0$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2$

$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\,\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos\theta$$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1 \bar{z}_2$

$$= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\,\bar{z}_2) = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos\theta \qquad \text{where } \theta = |\arg z_2 - \arg z_1|$$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(xi) If a and b are real numbers and z_1, z_2 are complex numbers, then

$$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = \left(a^2 + b^2\right)\left(|z_1|^2 + |z_2|^2\right)$$

(xii) If $z_1, z_2 \ne 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xiii) Triangle Inequality. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2| \le |z_1| + |z_2|$

(xiv) $|z_1 - z_2| \le |z_1| + |z_2|$

(xv) $\big||z_1| - |z_2|\big| \le |z_1| + |z_2|$

(xvi) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

POLAR FORM OF A COMPLEX NUMBER

Let z be a non-zero complex number, then we can write

$$z = r(\cos\theta + i\sin\theta) \qquad \text{where} \quad r = |z| \text{ and } \theta = \arg(z).$$

EULERIAN REPRESENTATION (EXPONENTIAL FORM)

Since we have, $e^{i\theta} = \cos\theta + i\sin\theta$

and thus z can be expressed as $z = re^{i\theta}$, where $|z| = r$ and $\theta = \arg(z)$

ARGUMENT OF A COMPLEX NUMBER

If z is a non – zero complex numbers represented by P in the complex plane, then argument of z is the angle which OP makes with positive direction of real axis.

Note : Argument of a complex number is not unique, since if θ is a value of the argument, then $2n\pi + \theta$ where $n \in I$, are also values of the argument of z. The value θ of the argument which satisfies the inequality $-\pi < \theta \le \pi$ is called the principal value of the argument or principal argument.

Thus, if $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0,\ y \geq 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0,\ y < 0 \end{cases}$$

PROPERTIES OF ARGUMENTS

(i) (a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ and $\arg(z_1 z_2 \ldots z_n) = \arg z_1 + \arg z_2 \ldots + \arg z_n$

(b) $\Pr\arg(z_1 z_2) = \Pr\arg(z_1) + \Pr\arg(z_2) + 2k\pi, \quad (k = 0 \text{ or } 1 \text{ or } -1)$

In general

$\Pr\arg(z_1 z_2 z_3 \ldots\ldots z_n) = \Pr\arg(z_1) + \Pr\arg(z_2) + \Pr\arg(z_3) + \ldots\ldots + \Pr\arg(z_n) + 2k\pi, \quad (k \in I)$

(ii) (a) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ and

(b) $\Pr\arg\left(\dfrac{z_1}{z_2}\right) = \Pr\arg z_1 - \Pr\arg z_2 + 2k\pi, \quad (k = 0 \text{ or } 1 \text{ or } -1)$

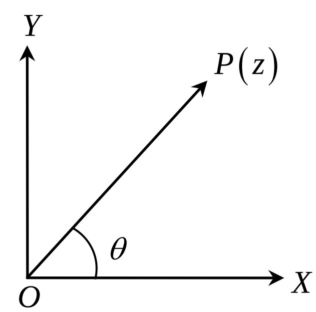
(iii) (a) $\arg z^n = n\arg z$,

(b) $\Pr\arg(z^n) = n\Pr\arg z + 2k\pi, \quad (k = 0 \text{ or } 1 \text{ or } -1)$

(iv) $\arg \bar{z} = -\arg z = \arg \dfrac{1}{z}$

VECTOR REPRESENTATION

If P is the point (x, y) on the argand plane corresponding to the complex number $z = x + iy$.



Then $\overrightarrow{OP} = x\hat{i} + y\hat{j}$,

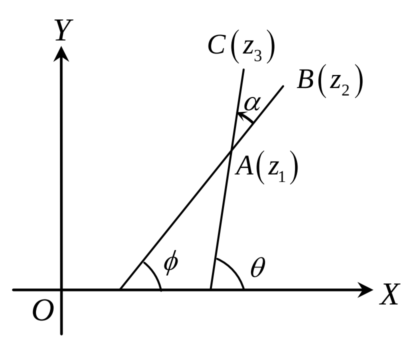
$\therefore \quad \left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2} = |z|$ and

$\theta = \arg(z) =$ direction of the vector $\overrightarrow{OP} = \tan^{-1}\left(\dfrac{y}{x}\right)$

ROTATION THEOREM

Rotational theorem i.e., angle between two intersecting lines. This is also known as coni formula.

Let z_1, z_2 and z_3 be the affixes of three points A, B and C respectively taken on argand plane.



Then we have $\overrightarrow{AC} = z_3 - z_1$

and $\overrightarrow{AB} = z_2 - z_1$

and let $\arg \overrightarrow{AC} = \arg(z_3 - z_1) = \theta$

and $\overrightarrow{AB} = \arg(z_2 - z_1) = \phi$

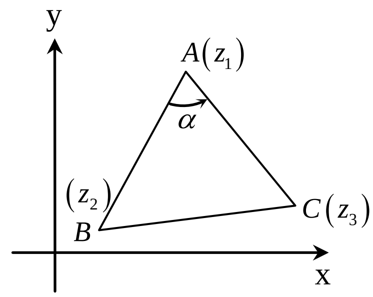
Let $\angle CAB = \alpha$,

$\because \quad \angle CAB = \alpha = \theta - \phi = \arg \overrightarrow{AC} - \arg \overrightarrow{AB}$

$$= \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

or angle between AC and AB = $\arg\left(\dfrac{\text{affix of } C - \text{ affix of } A}{\text{affix of } B - \text{ affix of } A}\right)$

Interpretation of arg $\left(\dfrac{\mathbf{z_3 - z_1}}{\mathbf{z_2 - z_1}}\right)$



If z_1, z_2, z_3 are the vertices of a triangle ABC described in the counter-clockwise sense, then

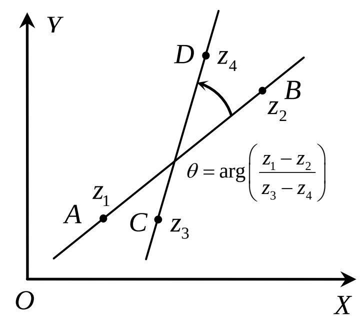
(i) $\arg\left(\dfrac{\overrightarrow{AC}}{\overrightarrow{AB}}\right) = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \angle BAC = \alpha$ (say), and

(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{CA}{BA}(\cos\alpha + i\sin\alpha)$

Corollary : The points z_1, z_2, z_3 will be collinear if and only if angle $\alpha = 0$ or π,

i.e., if and only if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real.

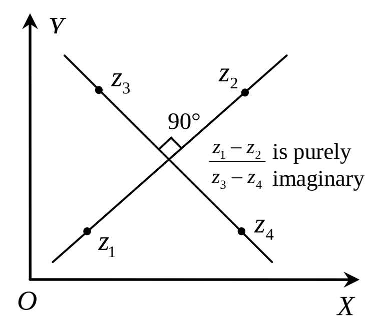
Interpretation of arg $\left(\dfrac{\mathbf{z_1 - z_2}}{\mathbf{z_3 - z_4}}\right)$



Let z_1, z_2, z_3 and z_4 be four complex numbers. Then the line joining z_4 and z_3 is inclined to the line joining z_2 and z_1 at the following angle :

$$\arg\left(\frac{\overrightarrow{CD}}{\overrightarrow{AB}}\right) = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$$

This can be proved analogously to the above result.



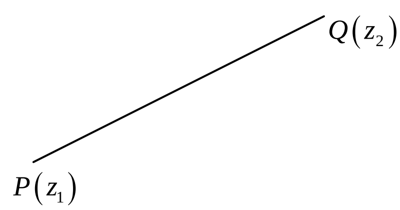
Corollary : The line joining z_4 and z_3 is inclined at $90°$ to the line joining z_2 and z_1 if

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

i.e., if $z_1 - z_2 = \pm ik(z_3 - z_4)$, where k is a non-zero real number.

SOME IMPORTANT GEOMETRICAL RESULTS AND EQUATIONS

1. **Distance Formula**

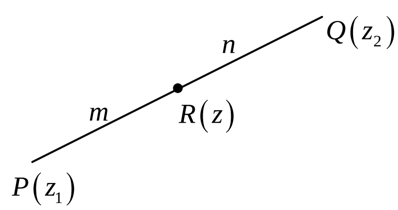


The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$$PQ = |z_2 - z_1| = |\textit{affix} \text{ of } Q - \textit{affix} \text{ of } P|$$

2. **Section Formula**

If *R(z)* divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m : n$ $(m, n > 0)$ then



(i) For internal division : $z = \dfrac{mz_2 + nz_1}{m + n}$

(ii) For external division : $z = \dfrac{mz_2 - nz_1}{m - n}$

3. **Condition (s) for four points *A* (z_1), *B*(z_2), *C*(z_3) and *D*(z_4) to represent vertices of *a***

(i) **Parallelogram**

The diagonals AC and BD must bisect each other

$$\Leftrightarrow \quad \frac{1}{2}(z_1+z_3)=\frac{1}{2}(z_2+z_4) \qquad \Leftrightarrow \qquad z_1+z_3=z_2+z_4$$

(ii) **Rhombus**

(a) the diagonals AC and BD bisect each other $\Leftrightarrow$ $z_1+z_3=z_2+z_4$, and

(b) a pair of two adjacent sides are equal, for instance, $AD=AB$ $\Leftrightarrow$ $|z_4-z_1|=|z_2-z_1|$

(c) $\arg\left(\dfrac{z_3-z_1}{z_4-z_2}\right)=\pm\dfrac{\pi}{2}$

(iii) **Rectangle**

(a) the diagonals AC and BD bisect each other $\Leftrightarrow$ $z_1+z_3=z_2+z_4$

(b) the diagonals AC and BD are equal $\Leftrightarrow$ $|z_3-z_1|=|z_4-z_2|$.

(iv) **Square**

(a) the diagonals AC and BD bisect each other $\Leftrightarrow$ $z_1+z_3=z_2+z_4$

(b) a pair of adjacent sides are equal; for instance $AD=AB$ $\Leftrightarrow$ $|z_4-z_1|=|z_2-z_1|$

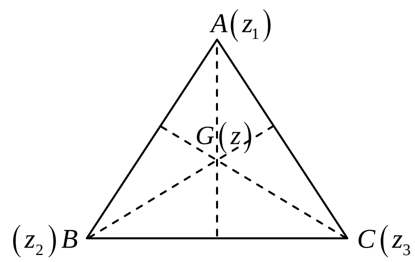
(c) the two diagonals are equal, that is, $AC=BD \Leftrightarrow |z_3-z_1|=|z_4-z_2|$

4. **Centroid, Incentre, Orthocentre and Circum-centre of a Triangle**

Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$,

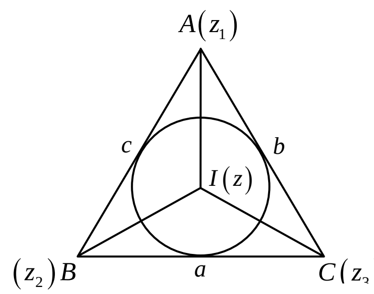
(i) **Centroid** $G(z)$ of the $\Delta\, ABC$ is given by

$$z=\frac{1}{3}(z_1+z_2+z_3)$$



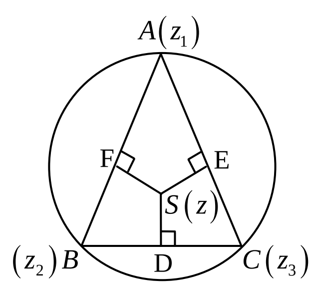
(ii) **Incentre** $I(z)$ of the $\Delta\, ABC$ is given by

$$z=\frac{az_1+bz_2+cz_3}{a+b+c}$$



(iii) **Circumcentre** $S(z)$ of the $\Delta\, ABC$ is given by

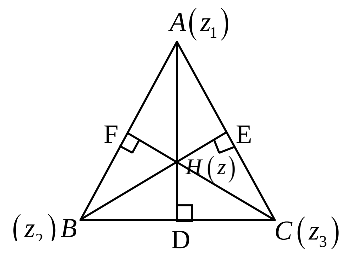
$$z=\frac{(z_2-z_3)|z_1|^2+(z_3-z_1)|z_2|^2+(z_1-z_2)|z_3|^2}{\bar{z}_1(z_2-z_3)+\bar{z}_2(z_3-z_1)+\bar{z}_3(z_1-z_2)} \quad \Leftrightarrow \quad z=\frac{\begin{vmatrix}|z_1|^2 & z_1 & 1\\ |z_2|^2 & z_2 & 1\\ |z_3|^2 & z_3 & 1\end{vmatrix}}{\begin{vmatrix}\bar{z}_1 & z_1 & 1\\ \bar{z}_2 & z_2 & 1\\ \bar{z}_3 & z_3 & 1\end{vmatrix}}$$



Also $\quad z=\dfrac{z_1(\sin 2A)+z_2(\sin 2B)+z_3(\sin 2C)}{\sin 2A+\sin 2B+\sin 2C}$

(iv) Orthocentre $H(z)$ of the ΔABC is given by

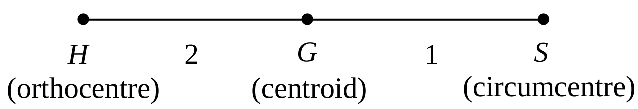
$$z = \frac{\begin{vmatrix} z_1^2 & \overline{z}_1 & 1 \\ z_2^2 & \overline{z}_2 & 1 \\ z_3^2 & \overline{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z}_1 & z_1 & 1 \\ \overline{z}_2 & z_2 & 1 \\ \overline{z}_3 & z_3 & 1 \end{vmatrix}}$$

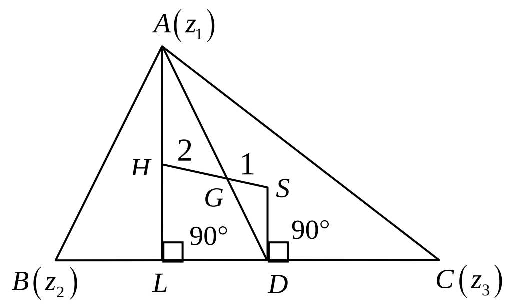


or $$z = \frac{(\tan A)z_1 + (\tan B)z_2 + (\tan C)z_3}{\tan A + \tan B + \tan C} = \frac{(a\sec A)z_1 + (b\sec B)z_2 + (c\sec C)z_3}{a\sec A + b\sec B + c\sec C}$$

Euler's Line

The centroid G of a triangle lies on the segment joining the orthocentre H and the circumcentre S of the triangle. G divides the join of H and S in the ratio $2:1$.





5. **Area of a Triangle**

Area of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \frac{1}{4} \text{ mod of } \left(i \begin{vmatrix} z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \\ z_3 & \overline{z}_3 & 1 \end{vmatrix} \right) = \frac{1}{4} \left| \begin{vmatrix} z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \\ z_3 & \overline{z}_3 & 1 \end{vmatrix} \right|$$

6. **Condition for Triangle to be Equilateral**

Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$$

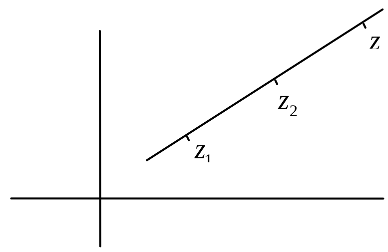
$$\Leftrightarrow \quad \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0$$

7. **Equation of a Straight Line**

(i) **Non-parametric form**

Equation of a straight line joining the points having affixes z_1 and z_2 is

$$\begin{vmatrix} z & \overline{z} & 1 \\ z_1 & \overline{z}_1 & 1 \\ z_2 & \overline{z}_2 & 1 \end{vmatrix} = 0$$



or $$\frac{z - z_1}{z_2 - z_1} = \frac{\overline{z} - \overline{z}_1}{\overline{z}_2 - \overline{z}_1}$$

or $\quad z\left(\bar{z}_1-\bar{z}_2\right)-\bar{z}\left(z_1-z_2\right)+z_1\bar{z}_2-z_2\bar{z}_1=0$

(ii) **Parametric form**

Equation of a straight line joining the points having affixes z_1 and z_2 is

$$z=tz_1+(1-t)z_2$$

where t is a real parameter.

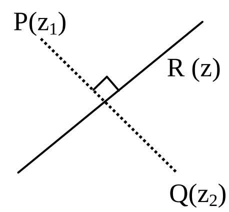
(iii) **General Equation of a Straight Line**

The general equation of a straight line is

$$\bar{a}z+a\bar{z}+b=0$$

where a is a complex number and b is a real number.

(iv) **Equation of the perpendicular bisector**



If $p(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is a moving point such that it is always at equal distance from $P(z_1)$ and $Q(z_2)$ then locus of $R(z)$ is perpendicular bisector of PQ

i.e. $\quad PR=QR$

or $\quad |z-z_1|=|z-z_2| \Rightarrow |z-z_1|^2=|z-z_2|^2$

after solving $z\left(\bar{z}_1-\bar{z}_2\right)+\bar{z}\left(z_1-z_2\right)=|z_1|^2-|z_2|^2$.

8. **Complex Slope of a Line**

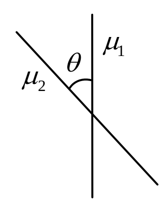
If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be

$$\mu=\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$$

Two lines with complex slopes μ_1 and μ_2 are

(i) parallel, if $\mu_1=\mu_2$

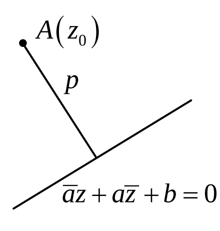
(ii) perpendicular, if $\mu_1+\mu_2=0$



(iii) angle θ between the lines is given by $\tan\theta=\left|\dfrac{i(\mu_2-\mu_1)}{\mu_2+\mu_1}\right|$

The complex slope of the line $\bar{a}z+a\bar{z}+b=0$ is given by $-(a/\bar{a})$

9. **Length of Perpendicular from a Point to a Line**



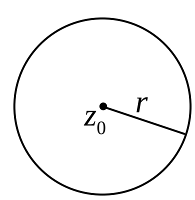
Length of perpendicular of point $A(z_0)$ from the line

$\bar{a}z+a\bar{z}+b=0 \qquad (a\in C,\, b\in R)$

is given by $\quad p=\dfrac{|\bar{a}z_0+a\bar{z}_0+b|}{2|a|}$

10. **Equation of Circle**

(i) An equation of the circle with centre at z_0 and radius r is



$$|z-z_0|=r$$

or $\quad z\bar{z}-z_0\bar{z}-\bar{z}_0z+z_0\bar{z}_0-r^2=0$

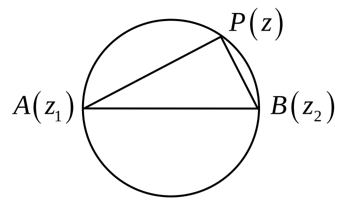
(ii) **General equation :**

General equation of a circle is $\quad z\bar{z}+a\bar{z}+\bar{a}z+b=0 \qquad \ldots (1)$

where a is a complex number and b is a real number.

Affix of centre of (1) is $-a$ and its radius is $\sqrt{a\bar{a}-b}$.

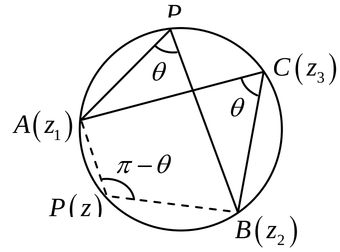
(iii) **Diametric From**



An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is

$(z-z_1)(\bar{z}-\bar{z}_2)+(\bar{z}-\bar{z}_1)(z-z_2)=0$

(iv) An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is



$$(z-z_1)(\bar{z}-\bar{z}_2)+(\bar{z}-\bar{z}_1)(z-z_2)+k\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0$$

where k is a real parameter.

(v) Equation of a circle passing through three non-collinear points.

Let three non -collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$.

Let $P(z)$ be any point on the circle.

Then either

$\angle ACB = \angle APB$ [when angles are in the same segment]

$\angle ACB + \angle APB = \pi$ [where angles are in the opposite segment]

$$\Rightarrow \quad \arg\left(\frac{z-z_2}{z-z_1}\right)-\arg\left(\frac{z_3-z_2}{z_3-z_1}\right)=0$$

$$\arg\left(\frac{z_3-z_2}{z_3-z_1}\right)+\arg\left(\frac{z-z_1}{z-z_2}\right)=\pi$$

$$\Rightarrow \quad \arg\left[\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z-z_1}{z-z_2}\right)\right]=0 \qquad \text{or}$$

$$\arg\left[\left(\frac{z-z_1}{z-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right)\right]=\pi$$

[using $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$ and $\arg(z_1\ z_2)=\arg(z_1)+\arg(z_2)$]

In any case, we get $\frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}$ is purely real.

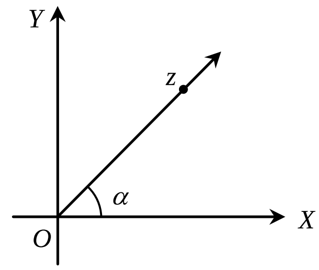
$$\Rightarrow \quad \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)}=\frac{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}$$

(vi) Condition for four points to be concyclic. Four points z_1, z_2, z_3 and z_4 will lie on the same circle if $\frac{(z_4-z_1)(z_3-z_2)}{(z_4-z_2)(z_3-z_1)}$ is purely real but $\frac{z_4-z_1}{z_4-z_2}$ is non-real

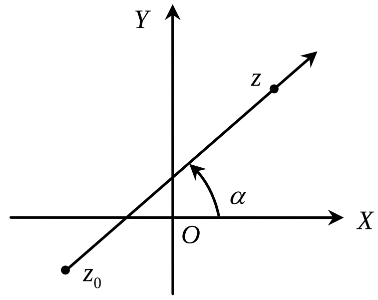
$\Leftrightarrow \quad \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real. But $\frac{z_1-z_3}{z_1-z_4}$ is non-real.

RECOGNIZING SOME LOCI BY INSPECTION

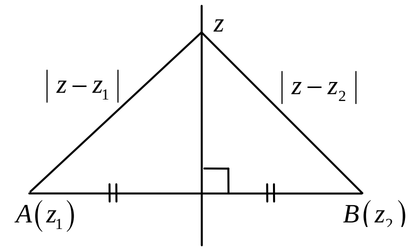
(i) If α is a real number, then $\arg(z)=\alpha$ represents a ray starting at the origin (excluding origin) and making an angle α with the real axis.



(ii) If α is a real number and z_0 is a fixed point,
then $\arg(z-z_0)=\alpha$
represents a ray starting at z_0 (excluding z_0) and making an angle α with the real axis.



(iii) If z_1 and z_2 are two fixed points,
then $|z-z_1|=|z-z_2|$
represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.



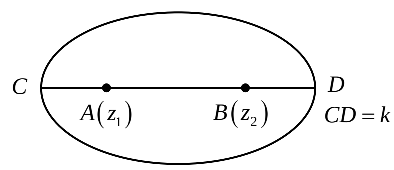
(iv) If z_1 and z_2 are two fixed points, and $k>0$, $k\neq 1$ is a real number, then

$$\frac{|z-z_1|}{|z-z_2|}=k$$

represents a circle.
For $k=1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

(v) Let z_1 and z_2 be two fixed points and k be a positive real number.

(a) If $k>|z_1-z_2|$, then
$|z-z_1|+|z-z_2|=k$
represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of major axis $=k$.

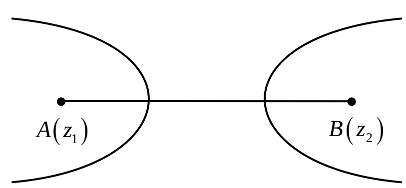


(b) If $k=|z_1-z_2|$, then $|z-z_1|+|z-z_2|=k$ represents the segment joining z_1 and z_2.

(c) If $k<|z_1-z_2|$, then $|z-z_1|+|z-z_2|=k$ does not represent any curve in the argand plane.

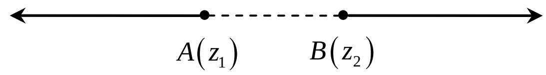
(vi) Let z_1 and z_2 be two fixed points, k be a positive real number.

(a) If $k<|z_1-z_2|$, then $\big||z-z_1|-|z-z_2|\big|=k$
represents a hyperbola with foci at $A(z_1)$ and $B(z_2)$.



(b) If $k = |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$

represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB.

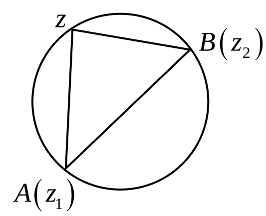


(c) If $k > |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$ does not represent any curve in Argand diagram.

(vii) If z_1 and z_2 represent two fixed points, then

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

represent a circle with z_1 and z_2 as extremities of a diameter.

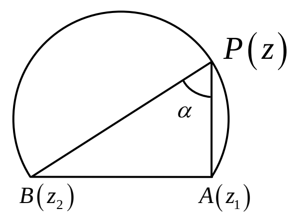


(viii) Let z_1 and z_2 be two fixed points, and α be a real number such $0 \le \alpha \le \pi$.

(a) If $0 < \alpha < \pi$

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$$

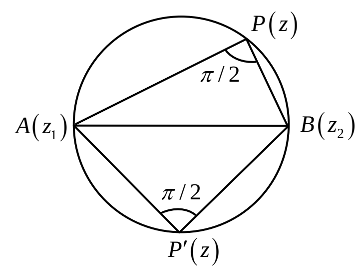
represents a segment of the circle passing through $A(z_1)$ and $B(z_2)$.



(b) If $\alpha = \pm\pi/2$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\left(= \pm\frac{\pi}{2}\right)$$

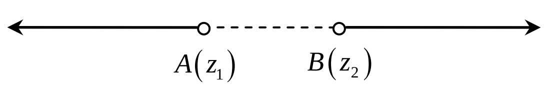
represents a circle with diameter as the segment joining $A(z_1)$ and $B(z_2)$ except the points A and B



(c) If $\alpha = 0$, then

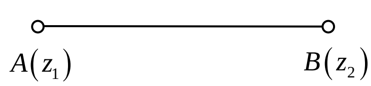
$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\,(= 0)$$

represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB.



(d) If $\alpha = \pi$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\,(= \pi)$$



represents the segment joining $A(z_1)$ and $B(z_2)$.

SQUARE ROOT OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number, Then $(a+ib)^{\frac{1}{2}} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right]$, for $b > 0$

$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right]$, for $b < 0$.

GEOMETRIC INTERPRETATION OF MULTIPLYING A COMPLEX NUMBER BY $e^{i\alpha}$.

Let z be a non-zero complex number. We can write z in the polar form as follows :

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

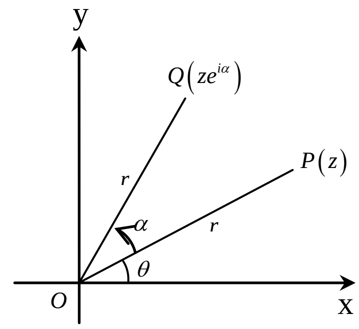
where $r = |\,z\,|$ and $\arg(z) = \theta$

We have

$$ze^{i\alpha} = re^{i\theta}\; e^{i\alpha} = re^{i(\theta+\alpha)}$$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$.

Geometrically, $ze^{i\alpha}$ can be obtained by rotating segment joining O and $P(z)$ through an angle α in the anticlockwise direction.



Corollary : If $A(z_1)$ and $B(z_2)$ are two complex number such that angle $\angle AOB = \theta$, then we can write

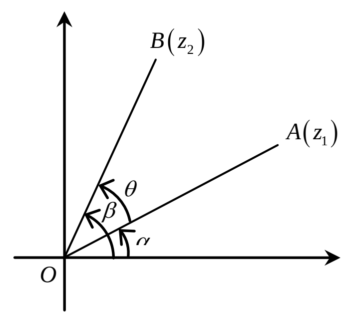
$$z_2 = \frac{|\,z_2\,|}{|\,z_1\,|} z_1 e^{i\theta}$$

Suppose $z_1 = r_1 e^{i\alpha}$ and $z_2 = r_2 e^{i\beta}$, where $|\,z_1\,| = r_1$, $|\,z_2\,| = r_2$. Then

$$\frac{z_2}{z_1} = \frac{r_2 e^{i\beta}}{r_1 e^{i\alpha}} = \frac{r_2}{r_1} e^{i(\beta-\alpha)}$$

Note that $\beta - \alpha = \theta$

Thus, $\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i\theta} \quad \Rightarrow \quad z_2 = \frac{|\,z_2\,|}{|\,z_1\,|} z_1 e^{i\theta}$



De Moivre's THEOREM AND ITS APPLICATIONS

(a) De Moivre's Theorem for integral index. If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(b) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$ where $\theta_1, \theta_2, \theta_3 \ldots \theta_n \in R$.

(c) De Moivre's Theorem for rational index.

If n is a rational, then one of the value of $(\cos\theta + i\sin\theta)^n$ is $\cos(n\theta) + i\sin(n\theta)$. In fact, if $n = p/q$ where $p, q \in I$, $q > 0$ and p, q have no factors in common, then $(\cos\theta + i\sin\theta)^n$ has q distinct values, one of which is $\cos(n\theta) + i\sin(n\theta)$.

In fact the values of $(\cos\theta + i\sin\theta)^{p/q}$ where $p, q \in I$, $q \neq 0$, hcf $(p, q) = 1$ are given by

$$\cos\left[\frac{p}{q}(2k\pi + \theta)\right] + i\sin\left[\frac{p}{q}(2k\pi + \theta)\right] \text{ where } k = 0, 1, 2, \ldots, q-1$$

(d) If n is irrational then $(\cos\theta + \sin\theta)^n$ has infinite number of values one of which is $\cos n\theta + i\sin n\theta$

The *n*th Roots of Unity

By an nth root of unity we mean any complex number z which satisfies the equation

$$z^n = 1 \qquad (1)$$

Since, an equation of degree n has n roots, there are n values of z which satisfy the equation (1). To obtain these n values of z, we put

$$1 = \cos(2k\pi) + i\sin(2k\pi)$$

where $k \in I$ and

$$\Rightarrow \qquad z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2\,k\pi}{n}\right) \qquad \text{[using De Moivre's Theorem]}$$

where $k = 0, 1, 2, ..., n-1.$

Let us put $\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

By using the De Moivre's theorem, we get the n th roots of unity as

$$1, \alpha, \alpha^2, ..., \alpha^{n-1}.$$

Sum of the Roots of Unity is Zero

We have

$$1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha}$$

But $\alpha^n = 1$ as α is a nth root of unity.

$\therefore \quad 1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$

Note

$$\frac{1}{x-1} + \frac{1}{x-\alpha} + \frac{1}{x-\alpha^2} + ... + \frac{1}{x-\alpha^{n-1}} = \frac{nx^{n-1}}{x^n - 1}$$

Product of n^{th} roots of unity is $(-1)^{n-1}$

Writing *n*th Roots of Unity When *n* is Odd

If $n = 2m+1,$ then nth roots of unity are also given by

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = -m, -(m-1), ..., -1, \quad 0, 1, 2, ... m.$

Since $\cos\left(-\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right)$

and $\sin\left(-\frac{2k\pi}{n}\right) = -\sin\left(\frac{2k\pi}{n}\right)$

we may take the roots as $1, \cos\left(\frac{2k\pi}{n}\right) \pm i\sin\left(\frac{2k\pi}{n}\right)$

where $k = 1, 2, ..., m.$

In terms of α we may take nth roots of unity to be $1, \alpha^{\pm 1}, \alpha^{\pm 2}, ... \alpha^{\pm m}$ or $1, \alpha^1, \bar{\alpha}^1, \alpha^2, \bar{\alpha}^2, \alpha^3, \bar{\alpha}^3 .... \alpha^m, \bar{\alpha}^n$

In this case, we may write $x^n - 1 = (x-1)(x-\alpha)(x-\alpha^{-1})(x-\alpha^2)(x-\alpha^{-2}) \ldots (x-\alpha^m)(x-\alpha^{-m})$ or $(x-1)(x-\alpha)(x-\bar{\alpha})(x-\alpha^2)(x-\bar{\alpha}^2).....(x-\alpha^m)(x-\bar{\alpha}^m)$

Writing *n*th Roots of Unity When *n* is Even

If $n = 2m$, then nth roots of unity are given $z = \pm 1, \pm\alpha, \pm\alpha^2, ... \pm\alpha^{m-1}$

Where $\alpha = \cos\left(\frac{2\pi}{2m}\right) + i\sin\left(\frac{2\pi}{2m}\right) = \cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right)$ or $\pm 1, \alpha, (\bar{\alpha}), \alpha^2, (\bar{\alpha})^2, \alpha^3, (\bar{\alpha})^3, ...\alpha^{m-1}, (\bar{\alpha})^{m-1}$

In this case, we may write

$x^n - 1 = (x-1)(x+1)(x-\alpha)(x+\alpha)(x-\alpha^2)(x+\alpha^2) \ldots (x-\alpha^{m-1})(x+\alpha^{m-1})$ or we may also write $(x-1)(x+1)(x-\alpha)(x-\bar{\alpha})(x-\alpha^2)(x-\bar{\alpha}^2) ...... (x-\alpha^{m-1})(x-\bar{\alpha}^{m-1})$

Cube Roots of Unity

Cube roots of unity are given by $1, \omega, \omega^2$

where $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Results Involving Complex Cube Root of Unity (ω)

(i) $\omega^3 = 1$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$

(iv) ω and ω^2 are roots of $x^2 + x + 1 = 0$

(v) $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

(vi) $a^2 + b^2 + c^2 - bc - ca - ab = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(vii) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(viii) $x^3 + 1 = (x+1)(x+\omega)(x+\omega^2)$

(ix) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(x) $a^4 + a^2b^2 + b^4 = (a+bw)(a+bw^2)(a-bw)(a-bw^2)$

(xi) Cube roots of real number a are $\sqrt[3]{a}, \sqrt[3]{a}\,w, \sqrt[3]{a}\,w^2$

To obtain cube roots of a, we write $x^3 = a$ as $y^3 = 1$ where $y = x/a^{1/3}$.

Solution of $y^3 = 1$ are $1, \omega, \omega^2$.

$\therefore \quad x = \sqrt[3]{a}, \sqrt[3]{a}\,w, \sqrt[3]{a}\,w^2$

***n*th Roots of a Complex Number**

Let $z \neq 0$ be a complex number. We can write z in the polar form as follows :

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \operatorname{Pr}\arg(z)$. Recall $-\pi < \theta \leq \pi$.

The nth root of z has n values one of which is equal to

$z_0 = \sqrt[n]{|z|}\left[\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right]$ and is called as the principal value of $z^{\frac{1}{n}}$. To obtain other value of $z^{\frac{1}{n}}$, we write z as

$$z = r\left[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right] \text{ where } k = 0, 1, 2 \ldots\ldots n-1$$

$$\Rightarrow \quad z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k$$

$$= z_0\ \alpha^k \text{ where } k = 0, 1, 2, \ldots, n-1$$

and $\alpha = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$ is a complex nth root of unity and $z_0 = \sqrt[n]{r}\left(\cos\dfrac{\theta}{n} + i\sin\dfrac{\theta}{n}\right)$.

Thus, all the n nth roots of z can be obtained by multiplying the principal value of $z^{\frac{1}{n}}$ by different roots of unity.

Rational Power of a Complex Number

If z is a complex number and m/n is a rational number such that m and n are relatively prime integers and $n > 0$.

Thus, $z^{m/n}$ has n distinct values which are given by

$$z^{m/n} = \left(\sqrt[n]{|z|}\right)^m\left[\cos\left(\frac{m}{n}(\theta + 2k\pi)\right) + i\sin\left(\frac{m}{n}(\theta + 2k\pi)\right)\right] \text{ where } k = 0, 1, 2, \ldots, n-1.$$

Logarithm of a Complex Number

Let $z \neq 0$ be a complex number. We may write z in the polar form as follows :

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

We define

$$\log z = \log r + i\theta$$

$$= \log|z| + i\,\mathrm{Pr}\arg(z)$$

$$z = x + iy, \text{ then}$$

$$\log z = \frac{1}{2}\log\left(x^2 + y^2\right) + i\,\mathrm{Pr}\arg(z).$$

Quadratic Equations **Chapter 4**

DEFINITIONS AND RESULTS

Real Polynomial : Let $a_0, a_1, a_2, a_3, \ldots a_n$ be real numbers and x is a real variable, then $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ is called a real polynomial of real variable x with real coefficients *e.g.* $f(x) = 2x^3 + 5x^2 - 7x - 9$, $f(x) = x^2 + 7x + 2$.

Complex Polynomial : If $a_0, a_1, a_2, \ldots a_n$ be complex numbers and x is complex variable, then $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

Degree of a Polynomial : The highest exponent of the variable of a polynomial is called the degree of the polynomial**,** Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \left(a_0 \neq 0\right)$. Then degree of the polynomial is n, where n is positive integer .

Polynomial Equation : An equation of the form $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0 \left(a_0 \neq 0\right)$. $a_0, a_1, \ldots a_n \in C$ is called a polynomial equation of degree n, an equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in C$ and $a \neq 0$ is called a polynomial equation of degree 2. A polynomial equation of degree 2 is called a quadratic equation. Similarly a polynomial equation of degree 3, degree 4 are respectively known as cubic equation, biquadratic equation.

IDENTITY

Identity is a statement of equality between two expressions which is true for all values of variable present in the equation *i.e.* $(x+6)^2 - 12x = x^2 + 36$ is an identity.

EQUATION

An equation is a statement of equality which is not true for all values of the variable present in the equation.

ROOTS OF AN EQUATION

The values of the variable which satisfies the given polynomial equation is called its roots *i.e. if* $f(x) = 0$ is a polynomial equation and $f(\alpha) = 0$, then α is known as root of the equation $f(x) = 0$.

ROOT OF THE QUADRATIC EQUATION

A quadratic equation $ax^2 + bx + c = 0$ has two and only two roots. The roots of $ax^2 + bx + c = 0$ are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

If α and β are roots of $ax^2 + bx + c = 0$, then

1. the sum of roots $= \alpha + \beta = \dfrac{-b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$
2. the product of roots $= \alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Also if $f(x) = ax^2 + bx + c = 0$, then $f(x) = a(x - \alpha)(x - \beta)$.

DISCRIMINANT OF A QUADRATIC EQUATION

If $ax^2 + bx + c = 0$, $a, b, c, \in R$ and $a \neq 0$, be a quadratic equation then the quantity $b^2 - 4ac$ is called discriminant of the equation and is generally denoted by D.

$\therefore \quad D = b^2 - 4ac$.

NATURE OF ROOTS OF THE QUADRATIC EQUATION:

For a quadratic equation $ax^2 + bx + c = 0$

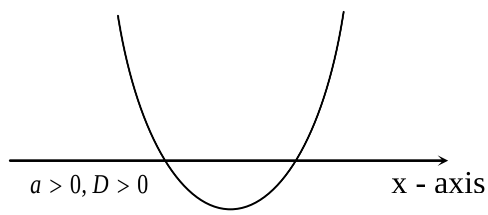
If $a, b, c, \in R$ and $a \neq 0,$ then

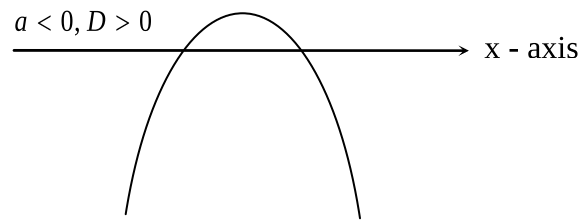
1. if $D > 0$ roots are real and distinct.
2. if $D = 0$ roots will be real and equal.
3. if $D < 0$ roots are imaginary and conjugate of each other.
4. if D is perfect square $a, b, c \in Q,$ then roots will be rational.
5. if D is not a perfect square roots will be irrational and conjugate of each other. If one root is $p + \sqrt{q}$ then other root be $p - \sqrt{q}$.
6. if $ax^2 + bx + c = 0$ has more than two roots (complex number or real number) then this is an identity *i.e.* $a = b = c = 0$.

GRAPH OF A QUADRATIC EXPRESSION

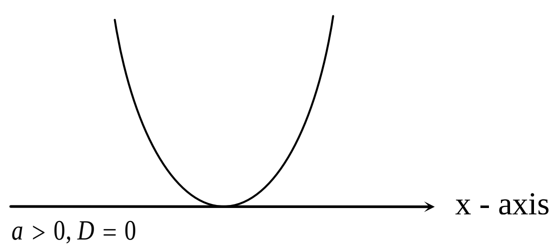
We have, $y = f(x) = ax^2 + bx + c$ where $a, b, c, \in R, a \neq 0.$

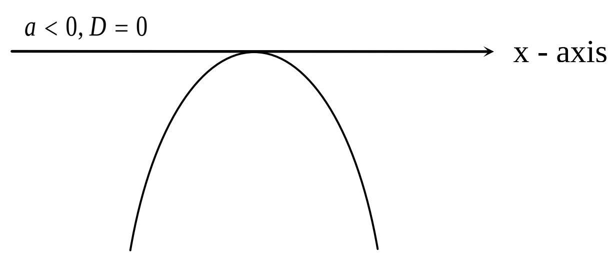
1. The shape of the curve $y = f(x)$ is a parabola
2. The axis of the parabola is parallel to y-axis.
3. If $a > 0,$ then the parabola opens upwards.
4. If $a < 0,$ then the parabola opens downwards
5. For $D > 0,$ parabola cuts x-axis in two distinct points





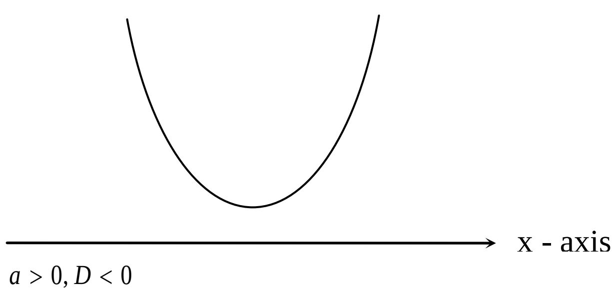
6. For $D = 0,$ parabola touches x-axis in one point.

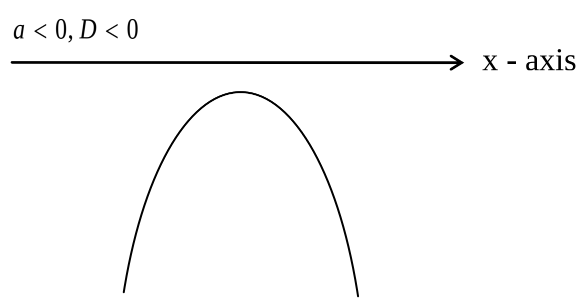




The co-ordinates of vertex of parabola in all these cases is $\left(\frac{-b}{2a}, \frac{-D}{4a} \right)$

7. For $D < 0,$ parabola does not cut x-axis.





GREATEST AND LEAST VALUES OF A QUADRATIC EXPRESSION

1. If $a > 0$, then the quadratic expression $ax^2 + bx + c$ has no greatest value but it has least value

$$\frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}.$$

2. If $a < 0$, then the quadratic expression $ax^2 + bx + c$ has no least value but it has greatest value

$$\frac{4ac - b^2}{4a} \text{ at } x = -\frac{b}{2a}.$$

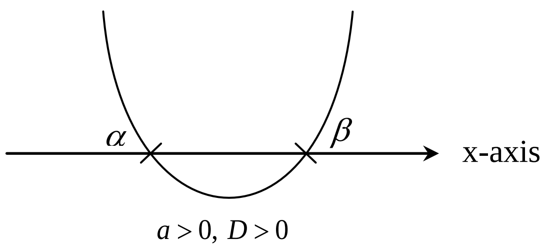
SIGN OF QUADRATIC EXPRESSION ax^2+bx+c

If α, β are roots of the corresponding quadratic equation, then for $x = \alpha$ and $x = \beta$ the value of the expression $ax^2 + bx + c$ is equal to zero. For other real value of *x*, the expression $ax^2 + bx + c > 0$ or < 0.

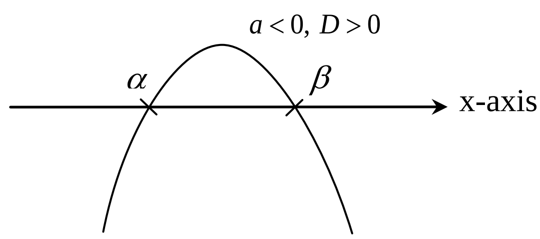
The sign of $ax^2 + bx + c, x \in R$ is determined by the following rule:

1. If $\alpha, \beta (\alpha < \beta)$ are real and unequal $(i.e.\ D > 0)$ roots of the corresponding quadratic equation then the sign of $y = ax^2 + bx + c, x \in R$ is determined as follows :

Clearly y is +ve for $x < \alpha$ or $x > \beta$ and y is −ve for $\alpha < x < \beta$

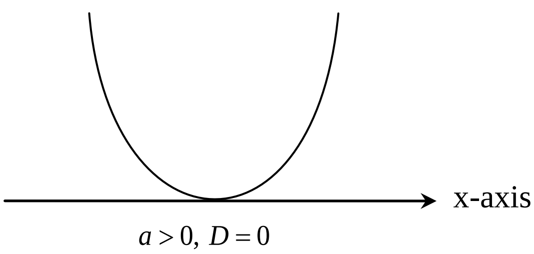


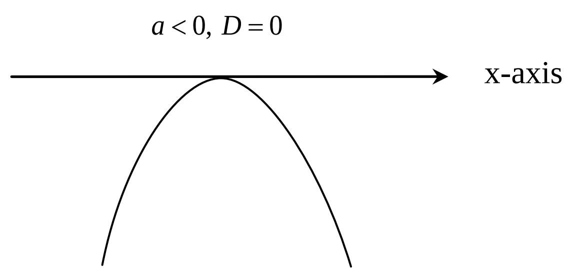
Clearly y is +ve for $\alpha < x < \beta$ and y is −ve for $x < \alpha$ and $x > \beta$



2. If α, β are real and equal $(i.e.\ D = 0)$ roots of the corresponding quadratic equation then the sign of $y = ax^2 + bx + c, x \in R$ is as follows:

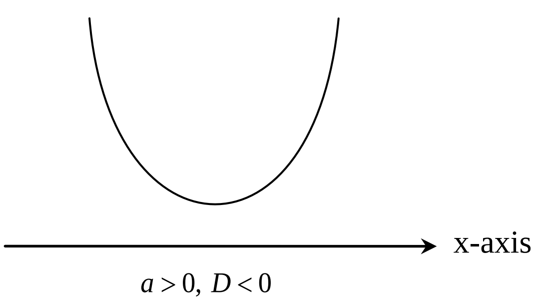
$y = ax^2 + bx + c \geq 0$ if $a > 0$ and $y = ax^2 + bx + c \leq 0$ if $a < 0$

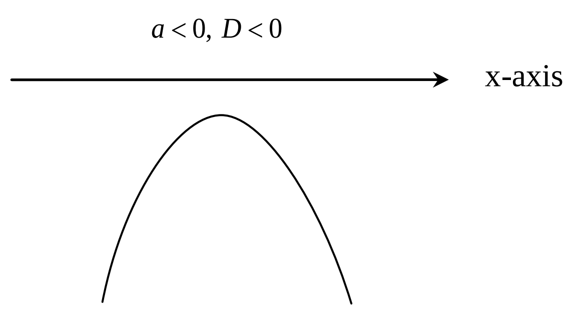




3. If α and β are imaginary $(i.e.\ D < 0)$, then

$y = ax^2 + bx + c > 0$ if $a > 0$ and $y = ax^2 + bx + c < 0$ if $a < 0$.





POSITION OF ROOTS OF A QUADRATIC EQUATION

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ be a quadratic expression.

Conditions for Both the Roots to be More Than a Real Number *k*.

Case I If $a > 0$, then the parabola $y = ax^2 + bx + c$ open upwards and intersect the x-axis in α and β where

$$\alpha = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

In this case both the roots α and β will be more than k if k lies to left of both α and β.

From the Fig I, we note that both the roots are more than k if and only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $f(k) > 0$

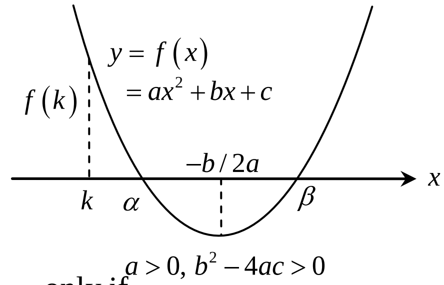


Fig. I

Case II In case $a < 0$, both the roots will be more than k if and only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $f(k) < 0$

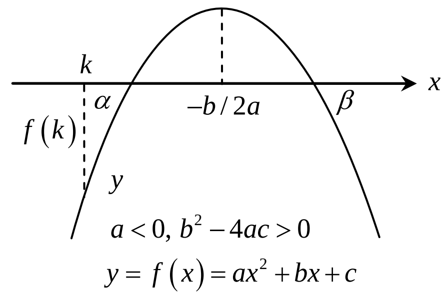


Fig. II

Combining the above two sets, we get both the roots of $ax^2 + bx + c = 0$ are more than a real number k if only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $af(k) > 0$

Condition for Both the Roots to be Less Than a Real Number *k*

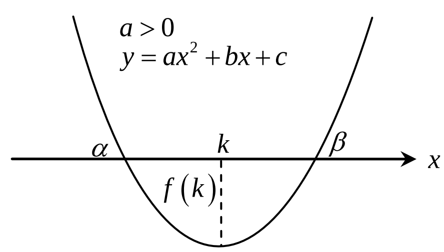
Both the roots of $ax^2 + bx + c = 0$ are less than a real number k if and only if

(i) $D > 0$ (ii) $k > \frac{-b}{2a}$ (iii) $af(k) > 0$

CONDITIONS FOR A NUMBER *k* TO LIE BETWEEN THE ROOTS OF A QUADRATIC EQUATION

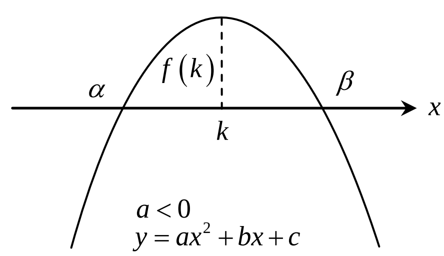
The real number k lies between the roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$ if and only if a and $f(k)$ are of opposite signs, that is, if and only if

(i) $a > 0$ (ii) $f(k) < 0$



Note that $D > 0$ is surely true if $f(k) < 0$

(i) $a < 0$ (ii) $f(k) > 0$



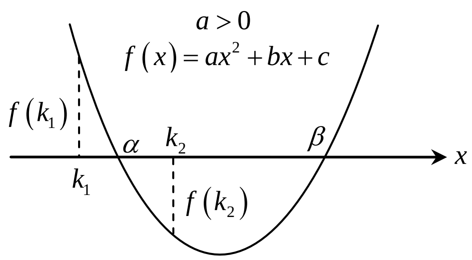
Note that $D > 0$ is surely true if $f(k) > 0$

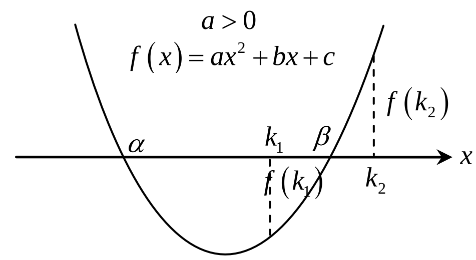
Combining, we may say k lies between the roots of $f(x) = ax^2 + bx + c = 0$ if and only if $af(k) < 0$

CONDITIONS FOR EXACTLY ONE ROOT OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL *(k₁, k₂)* WHERE *k₁ < k₂*

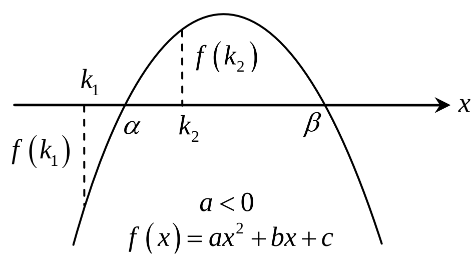
Case I If $a > 0$, then exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1) > 0$ and $f(k_2) < 0$ or $f(k_1) < 0$ and $f(k_2) > 0$.

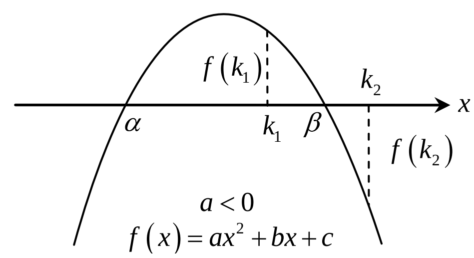
Thus, if $a > 0$, exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1)\, f(k_2) < 0$.





Case II Similarly, if $a < 0$, exactly one of the roots of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if $f(k_1)\, f(k_2) < 0$.



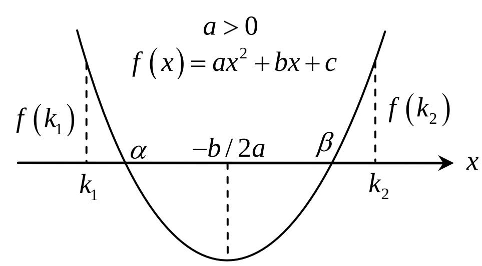


CONDITIONS FOR BOTH THE ROOTS OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL *(k₁, k₂)* WHERE *k₁ < k₂*.

Case I If $a > 0$, both the roots of $f(x) = ax^2 + bx + c = 0$ lie in the interval (k_1, k_2) if and only if

(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

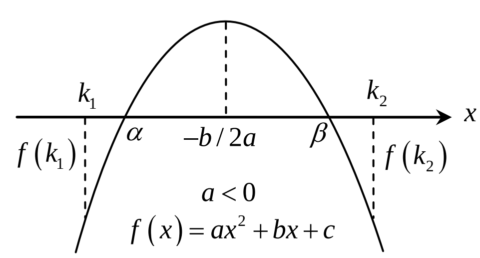
(iii) $f(k_1) > 0$ and $f(k_2) > 0$



Case II If $a < 0$, the conditions are

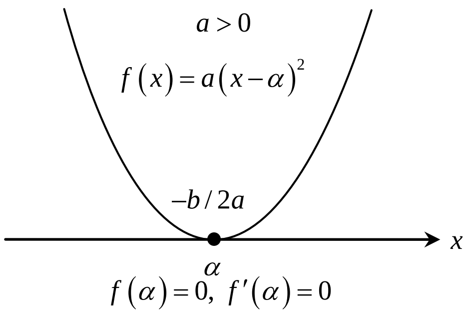
(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

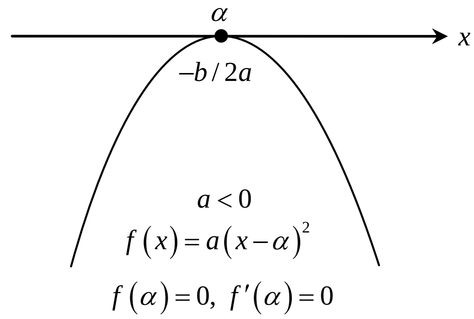
(iii) $f(k_1) < 0$ and $f(k_2) < 0$



CONDITIONS FOR A QUADRATIC EQUATION TO HAVE A REPEATED ROOT

The quadratic equation $f(x) = ax^2 + bx + c = 0,\ a \neq 0$ has α as a repeated if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0.$ In this case $f(x) = a(x-\alpha)^2$. In fact $\alpha = -b/2a.$





CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Suppose that the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ (where $a_1, a_2 \neq 0$ and $a_1b_2 - a_2b_1 \neq 0$) have a common root. Let α be the common root.

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$

Solving the above equations, we get

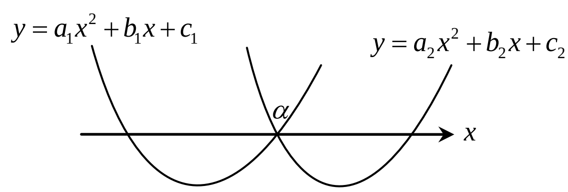
These give

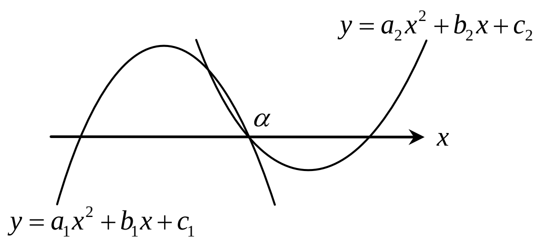
$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

Eliminating α we get

$$(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$$

This is the required condition for two quadratic equation to have a common root.





CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE THE SAME ROOTS (i.e. Both root common)

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

In this case both the equations have the same roots.

Note :

(a) To find the common root of two equations, make the coefficient of second degree terms in two equations equal and subtract. The value of *x* so obtained is the required common root.

(b) If two quadratic equations with real coefficients have an imaginary root common, then both roots will be common and the two equations will be identical. $\therefore\ \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}.$

(c) If two quadratic equations with rational coefficients have an irrational root common, then both roots will be common and the two equations will be identical. $\therefore\ \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}.$

(d) If α is a repeated root of the quadratic equation $f(x) = ax^2 + bx + c = 0$, then α is also a root of the equation $f'(x) = 0$.

HIGHER DEGREE EQUATIONS

The equation $f(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ … (1)

Where the coefficient $a_0, a_1........a_n \in C$ and $a_0 \neq 0$ is called an equation of n^{th} degree, which has exactly n roots $\alpha_1, \alpha_2......\alpha_n \in C$, Then we can write

$$p(x) = a_0(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n) = a_0\left\{x^n - \left(\sum\alpha_1\right)x^{n-1} + \left(\sum\alpha_1\alpha_2\right)x^{n-2} - ...... + (-1)^n \alpha_1\alpha_2.....\alpha_n\right\} \quad \text{… (2)}$$

Comparing (1) and (2) $\sum\alpha_1 = \alpha_1 + \alpha_2 + ... + \alpha_n = \dfrac{-a_1}{a_0}$; $\sum\alpha_1\alpha_2 = \alpha_1\alpha_2 + ... + \alpha_{n-1}\alpha_n = \dfrac{a_2}{a_0}$

And so on and $\alpha_1\alpha_2...\alpha_n = (-1)^n \dfrac{a_n}{a_0}$

CUBIC EQUATION

For $n = 3$, the equation is a cubic of the form $ax^3 + bx^2 + cx + d = 0$ and we have in this case $\alpha + \beta + \gamma = -\dfrac{b}{a}$; $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$; $\alpha\beta\gamma = -\dfrac{d}{a}$

BIQUADRATIC EQUATION

If $\alpha, \beta, \gamma, \delta$ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$\sigma_1 = \alpha + \beta + \gamma + \delta = -\dfrac{b}{a}$, $\sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \dfrac{c}{a}$

$\sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\dfrac{d}{a}$, $\sigma_4 = \alpha\beta\gamma\delta = \dfrac{e}{a}$.

TRANSFORMATION OF EQUATIONS

Rules to form an equation whose roots are given in terms of the roots of another equation. Let given equation be

$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ (1)

Rule 1 : To form an equation whose roots are $k(\neq 0)$ times roots of the equations in (1), replace x by x/k in (1).

Rule 2 : To form an equation whose roots are the negatives of the roots in equation (1), replace x by $-x$ in (1). Alternatively, change the sign of the coefficients of x^{n-1}, x^{n-3}, x^{n-5},... etc. in (1).

Rule 3 : To form an equation whose roots are k more than the roots of equation in (1), replace x by $x - k$ in (1).

Rule 4 : To form an equation whose roots are reciprocals of the roots in equation (1), replace x by $1/x$ in (1) and then multiply both the sides by x^n.

Rule 5 : To form an equation whose roots are square of the roots of the equation in (1) proceed as follows :

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are squares of the roots of $x^3 - 2x^2 - x + 2 = 0$, replace x by $\sqrt{x}$ to obtain

$$x\sqrt{x} - 2x - \sqrt{x} + 2 = 0 \quad \Rightarrow \quad \sqrt{x}(x-1) = 2(x-1)$$

Squaring we get

$$x(x-1)^2 = 4(x-1)^2$$

or $(x-4)(x^2-2x+1)=0$ or $x^3-6x^2+9x-4=0$

Rule 6 : To form an equation whose roots are cubes of the roots of the equation in (1) proceed as follows :

Step 1 Replace x by $x^{1/3}$.

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify

DESCARTES RULE OF SIGNS FOR THE ROOTS OF A POLYNOMIAL

Rule 1 : The maximum number of positive real roots of a polynomial equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

is the number of changes of the signs of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3+3x^2+7x-11=0$ the signs of coefficients are $+ + + -$ As there is just one change of sign, the number of positive roots of $x^3+3x^2+7x-11=0$ is at most 1.

Rule 2 : The maximum number of negative roots of the polynomial equation $f(x)=0$ is the number of changes from positive to negative and negative to positive in the signs of coefficients of the equation $f(-x)=0$.

RESULT ON ROOTS OF AN EQUATION

Let $f(x)=0$ is polynomial equation of degree n then

1. Number of real roots of the equation is less than or equal to the degree of the equation.
2. Total number of roots of the equation are :
   (i) Number of positive roots
   (ii) Number of negative roots
   (iii) Number of imaginary roots.
   (iv) Number of roots which are zero

   $\therefore$ Total number of roots = Number of positive roots + Number of negative roots + Number of complex roots + number of roots which are zero.

Note:

(a) If $f(x)$ = 0 be an equation and a and b are two real numbers such that $f(a)f(b)<0,$ then equation $f(x)=0$ has at least one real root or an odd number of real roots (may be repeated) between a and b.

(b) If $f(a)<0.\ f(b)<0\ or\ f(a)>0, f(b)>0,$ then either no real root or an even number of real roots (may be repeated) of $f(x)=0$ lies between the numbers a and b.

CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO RESOLVE INTO TWO LINEAR FACTORS

The general quadratic expression $ax^2+2hxy+by^2+2gx+2fy+c$ in x and y may be resolved into two linear factors if $abc+2fgh-af^2-bg^2-ch^2=0$

i.e. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Case I If $h^2-ab>0$ then the given expression can be resolved into two real linear factors.

Case II If $h^2 - ab < 0$ then the two factors are complex.
Case III If $h^2 - ab = 0$ then the two real factors are same.

RATIONAL ALGEBRAIC EXPRESSION

An expression of the form $\frac{P(x)}{Q(x)}$ (where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$) is known as a rational algebraic expression.

SOLUTION OF INEQUATION

If $a > 0$

1. $|x| < a$ or $x^2 < a^2 \Leftrightarrow -a < x < a$
2. $|x| > a$ or $x^2 > a^2 \Leftrightarrow x < -a \cup x > a$
   or $R - [-a, a]$
3. If $\alpha < \beta$
   $(x-\alpha)(x-\beta) < 0 \Leftrightarrow \alpha < x < \beta$
   and $(x-\alpha)(x-\beta) \leq 0 \Leftrightarrow \alpha \leq x \leq \beta$
4. If $\alpha < \beta$
   $(x-\alpha)(x-\beta) > 0 \Leftrightarrow x < \alpha \cup x > \beta$
   and $(x-\alpha)(x-\beta) \geq 0$
   $\Leftrightarrow x \leq \alpha \cup x \geq \beta$

The method of intervals (Wavy curve method)

This method is used for solving inequalities of type

Say $f(x) = \frac{g(x)}{p(x)} = \frac{(x-a_1)^{p_1}(x-a_2)^{p_2}.......(x-a_k)^{p_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}......(x-b_t)^{m_t}} > 0, p(x) \neq 0$

< 0

≥ 0

≤ 0

(i) where $a_{i,} b_j$ are real number $\begin{pmatrix} i = 1,2,3.....k \\ j = 1,2,3.....t \end{pmatrix}$ such that $a_i \neq b_j$

(ii) $p_1, p_2 ... p_k, m_1, m_2 ... \; ... m_t$ are natural numbers

(iii) $a_1, < a_2 < a_3 ... < a_k$ and $b_1 < b_2 < b_3 ... < b_t$

Steps to find the sign of the function in an interval

(a) All zeroes of $g(x)$ and $p(x)$ are marked on the number line.

(b) All zeroes of $p(x)$ are known as points of discontinuities

(c) $g(x)$and $p(x)$ should not contain common zeroes

(d) Even and odd powers of all factors of $g(x)$ and $p(x)$ should be noted.

(e) Now after marking of the numbers on the number line put positive sign $(+ve)$ in the right of the biggest of these number.

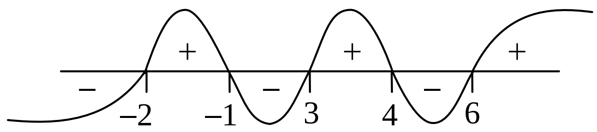
(f) When passing through the next point the polynomial changes its sign if its power is odd and polynomial have the same sign if its power is even and continue the process by the same rule.

(g) The solution of $f(x) > 0$ is the union of all those, intervals in which we have put + ve sign.

(h) The solution of $f(x)<0$ is the union of all those intervals in which we have put negative (minus) sign.

Example : (i) Let $f(x)=\dfrac{(x-3)^1(x+2)^1(x-4)^1}{(x+1)^1(x-6)^1}>0$

Critical Points $-2,-1,3,4,6$



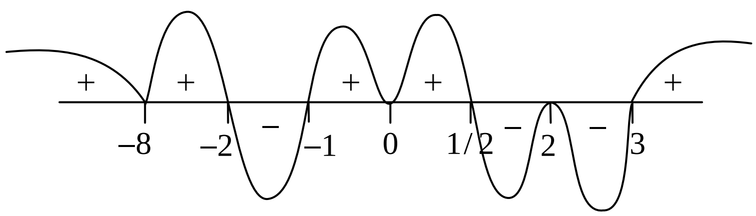
In our problem each and every factor occurs odd time so interval gets alternatively + and –sign

$\therefore \quad f(x)>0\,\forall x\in(-2,-1)\cup(3,4)\cup(6,\infty)$

$f(x)<0\,\forall x\in(-\infty,-2)\cup(-1,3)\cup(4,6)$

Example : (ii) Let $f(x)=\dfrac{(x-2)^{998}(x+1)^{249}(x-1/2)^{87}(x+8)^6}{x^{26}(x-3)^{41}(x+2)^{39}}$

Critical points are $-8,-2,-1,0,1/2,2,3$



EQUATIONS WHICH CAN BE REDUCED TO LINEAR, QUADRATIC AND BIQUADRATIC EQUATIONS

1. To solve an equation of the form $(x-a)^4+(x-b)^4=A$

   put $y=x-\dfrac{a+b}{2}$

   In general to solve an equation of the form $(x-a)^{2n}+(x-b)^{2n}=A$

   where n is a positive integer, we put $y=x-\dfrac{a+b}{2}$.

2. To solve an equation of the form $a_0 f(x)^{2n}+a_1(f(x))^n+a_2=0$ (1)

   we put $(f(x))^n=y$ and solve $a_0y^2+a_1\,y+a_2=0$ to obtain its roots y_1 and y_2.

   Finally, to obtain solutions of (1) we solve,

   $(f(x))^n=y_1$

   and $(f(x))^n=y_2$

3. An equation of the form

   $(ax^2+bx+c_1)(ax^2+bx+c_2)...(ax^2+bx+c_n)=A$

   where $c_1,\,c_2,...,c_n,\ A\in R,$ can be solved by putting $ax^2+bx=y$.

4. An equation of the form

$$(x-a)(x-b)(x-c)(x-d) = Ax^2$$

where $ab = cd$, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{x}$.

5. An equation of the form

$$(x-a)(x-b)(x-c)(x-d) = A$$

where $a < b < c < d$, $b - a = d - c$ can be solved by change of variable

$$y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4} \qquad = x - \frac{1}{4}(a+b+c+d)$$

6. A polynomial $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x)\ \forall\ x, y$.

A symmetric polynomials can be represented as a function of $x + y$ and xy.

USE OF CONTINUITY AND DIFFERENTIABILITY TO FIND ROOTS OF A POLYNOMIAL EQUATION

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$, then f is continuous on $\boldsymbol{R}$

Since f is continuous on $\boldsymbol{R}$, we may use the intermediate value theorem to find whether or not f has a real root. If there exists a and b such that $a < b$ and $f(a) f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$. Also, if $f(-\infty)$ and $f(a)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in $(-\infty, a)$. Also, if $f(a)$ and $f(\infty)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in (a, ∞).

Result 1 If $f(x) = 0$ has a root α of multiplicity r (where $r > 1$), then we can write

$$f(x) = (x - \alpha)^r g(x)$$

where $g(\alpha) \neq 0$.

Also, $f'(x) = 0$ has α as a root with multiplicity $r - 1$.

Result 2 If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has at least $(n-1)$ real roots.

It follows immediately using Result 1 and Rolle's Theorem.

Result 3 If $f(x) = 0$ has n distinct real roots, we can write

$$f(x) = a_0 (x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$$

where $\alpha_1, \alpha_2, ...\alpha_n$ are n distinct roots of $f(x) = 0$.

We can also write $\dfrac{f'(x)}{f(x)} = \sum_{k=1}^{n} \dfrac{1}{x - \alpha_k}$

Determinants Chapter 5

DEFINITION

Any $n \times n \; \left(= n^2\right)$ numbers arranged in a square form along horizontal rows and vertical columns having n elements in each row and column enclosed between two vertical bars is called a **determinant** of order n .

EVALUATION OF DETERMINANTS

A determinant of order two is written as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\qquad \left(a_{ij} \in C \forall i, j\right)$

and is equal to $\qquad a_{11}\, a_{22} - a_{12}\, a_{21}$.

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \left(a_{ij} \in C \forall i, j\right)$$

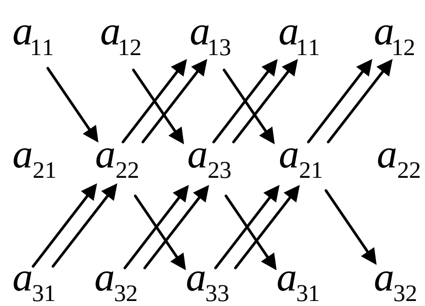
and is equal to

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{12}a_{21}a_{33}$$

A determinant of order 3 can also be evaluated by using the following diagram, due to Sarrus:



The product of the three terms on each of the three single arrows are prefixed by a positive sign and the product of the three terms on each of the three double arrows are prefixed by a negative sign.

The determinant obtained by deleting the i^{th} row and j^{th} column is denoted by M_{ij} and is called the *minor* of element a_{ij} . The *co-factor* of the element a_{ij} is denoted by C_{ij} and is given by $\left(-1\right)^{i+j} M_{ij}$.

Note that $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13} = a_{11}\, C_{11} + a_{12}\, C_{12} + a_{13}\, C_{13}$

If $\quad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $\quad \Delta = a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + a_{i3}\, C_{i3} \qquad i = 1, 2, 3$

$$= a_{1j}\, C_{1j} + a_{2j}\, C_{2j} + a_{3j}\, C_{3j} \qquad j = 1, 2, 3$$

Also $a_{i1}C_{j1}+a_{i2}C_{j2}+a_{i3}C_{j3}=0 \quad i\neq j$ and $a_{1i}C_{1j}+a_{2i}C_{2j}+a_{3i}C_{3j}=0 \quad i\neq j$

The above results remain true for determinants of every order.

PROPERTIES OF DETERMINANTS

1. **Reflection property :** The determinant remains unaltered if its rows are changed into columns and the columns into rows.
2. **All-zero property :** If all the elements of a row (column) are zero, then the determinant is zero.
3. **Proportionality [Repetition] Property :** If the elements of a row (columns) are proportional [identical] to the element of some other row (columns), then the determinant is zero.
4. **Switching Property :** The interchange of any two rows (columns) of the determinant changes its sign.
5. **Scalar Multiple property :** If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.
6. **Sum Property :** $\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$
7. **Property of Invariance :**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha a_2+\beta a_3 & b_1+\alpha b_2+\beta b_3 & c_1+\alpha c_2+\beta c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

That is , a determinant remains unaltered under an operation of the form

$C_i \to C_i+\alpha C_j+\beta C_k, \text{where } j,k\neq i,$ or an operation of the form

$R_i \to R_i+\alpha R_j+\beta R_k, \text{where } j,k\neq i,$

8. **Factor property :** If a determinant Δ becomes zero when we put $x=\alpha,$ then $(x-\alpha)$ is a factor of Δ.
9. **Triangle Property :** If all the elements of a determinant above or below the main diagonal consists of zeros, then the determinant is equal to the products of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

10. **Product of Two Determinants :**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

Here we have multiplied rows by rows. We can also multiply rows by columns, or columns by rows, or columns by columns.

11. **Conjugate of a Determinant :** If $a_i, b_i, c_i \in C\,(i=1,2,3)$,

and $Z=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then $\overline{Z}=\begin{vmatrix} \overline{a_1} & \overline{b_1} & \overline{c_1} \\ \overline{a_2} & \overline{b_2} & \overline{c_2} \\ \overline{a_3} & \overline{b_3} & \overline{c_3} \end{vmatrix}$

SUMMATION OF DETERMINANTS : Let $\Delta_r = \begin{vmatrix} f(r) & a & l \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$ where a, b, c, l, m and n are constants.

Then $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & l \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}$. Here function of r can be elements of only one row or one column.

DIFFERENTIATION OF A DETERMINANT

1. Let $\Delta(x)$ be a determinant of order two. If we write $\Delta x = \begin{vmatrix} C_1 & C_2 \end{vmatrix}$, where C_1 and C_2 denote the 1^{st} and 2^{nd} columns, then

$$\Delta'(x) = \begin{vmatrix} C_1' & C_2 \end{vmatrix} + \begin{vmatrix} C_1 & C_2' \end{vmatrix}$$

where C_i' denotes the column which contains the derivative of all the functions in the i^{th} column C_i.

In a similar fashion, if we write $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R_1' \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \end{vmatrix}$.

2. Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = \begin{vmatrix} C_1 & C_2 & C_3 \end{vmatrix}$, then

$$\Delta'(x) = \begin{vmatrix} C_1' & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} C_1 & C_2' & C_3 \end{vmatrix} + \begin{vmatrix} C_1 & C_2 & C_3' \end{vmatrix}$$

and similarly if we consider $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$. Then $\Delta'(x) = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix}$.

3. If only one row (or column) consists functions of *x* and other rows (or columns) are constant, *viz*. Let

$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Then $\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

and in general $\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

INTEGRATION OF A DETERMINANT

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$, where a, b, c, l, m and n are constants.

$$\Rightarrow \int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

Note :

If the elements of more than one column or rows are functions of *x* then the integration can be done only after evaluation/expansion of the determinant.

Note :

Result; Suppose $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then $\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$

where C_{ij} denotes the co-factor of the element a_{ij} in Δ.

SOME SPECIAL DETERMINANTS

1. **Symmetric determinant**

A determinant is called symmetric determinant if for its every elements

$$a_{ij} = a_{ji} \;\forall\; i,\, j \;\; e.g., \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

2. **Skew-symmetric determinant :** A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji} \;\forall\; i,\, j,\; e.g. \begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Note: Every diagonal element of a skew symmetric determinant is always zero.
The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

3. **Cyclic order :** If elements of the rows (or columns) are in cyclic order i.e.,

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$.

(iv) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

CRAMER'S RULE

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ then the solution of the system of liner equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta} \text{ where}$$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{and } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

CONDITIONS FOR CONSISTENCY

The following cases may arise :

1. If $\Delta \neq 0,$ then the system is consistent and has a unique solution, which is given by Cramer's rule :

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}.$$

2. If $\Delta = 0$ and atleast one of the determinants $\Delta_1, \Delta_2, \Delta_3$ is non-zero, the given system is inconsistent i.e. it has no solution.
3. If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0,$ then the system is consistent and dependent, and has infinitely many solutions.

HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEM

The system of linear homogeneous equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

has a non-trivial solution (i.e., at least one of the x, y, z is different from zero)

if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

If $\Delta \neq 0$, then the only solution of the above system of equations is $x = 0$, $y = 0$ and $z = 0$.

Corollary If at least one of *x, y, z,* is non-zero and *x, y* and *z* are connected by the three given equations, then the elimination of *x*, *y* and *z* leads to the relation $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Matrices
Chapter 6

DEFINITION

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. The numbers are called the elements of the matrix or entries in the matrix.

ORDER OF A MATRIX

A matrix having *m* rows and *n* columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an *m* by *n* matrix). A matrix A of order $m \times n$ is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & a_{23} & ...a_{2j} & ...a_{2n} \\ ... & ... & ... & ... & ... \\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & a_{in} \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & a_{m3} & ...a_{mj} & ...a_{mn} \end{bmatrix} \text{ or } A = \left[a_{ij}\right]_{m \times n}, \quad \text{where } \begin{matrix} i = 1, 2, ... m \\ j = 1, 2, ... n \end{matrix}$$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

Example : order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2×3.

A matrix of order $m \times n$ contains mn elements. Every row of such a matrix contains *n* elements and every column contains *m* elements.

EQUALITY OF MATRICES

Two matrices *A* and *B* are said to be equal matrix if they are of same order and their corresponding elements are equal.

TYPES OF MATRICES

1. **Row matrix :** A matrix is said to be a row matrix or row vector if it has only one row and any number of columns.

   Example : $\begin{bmatrix} 5 & 0 & 3 \end{bmatrix}$ is a row matrix of order 1×3 and [2] is a row matrix of order 1×1.

2. **Column matrix :** A matrix is said to be a column matrix or column vector if it has only one column and any number of rows.

   Example : $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3×1 and $[2]$ is a column matrix of order 1×1. Observe that $[2]$ is both a row matrix as well as a column matrix.

3. **Singleton matrix :** If in a matrix there is only one element then it is called singleton matrix. Thus, $A = \left[a_{ij}\right]_{m \times n}$ is a singleton matrix, if $m = n = 1$.

   Example : $[2], [3], [a], [-3]$ are singleton matrices.

4. **Null or zero matrix :** If in a matrix all the elements are zero then it is called zero matrix and it is generally denoted by *O*. Thus $A = \left[a_{ij}\right]_{m \times n}$ is a zero matrix if $a_{ij} = 0$ for all *i* and *j*.

Example : $[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [0 \quad 0]$ are all zero matrices, but of different orders.

5. **Square matrix :** If number of rows and number of columns in a matrix are equal, then it is called a square matrix. Thus $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.

Example : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3×3.

(i) If $m \neq n$ then matrix is called a rectangular matrix.

(ii) The elements of a square matrix A for which $i = j$, i.e., $a_{11}, a_{22}, a_{33}, ...a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix *A*.

6. **Diagonal matrix :** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix or order 3×3, which can be denoted by diag. $[2, \; 3, \; 4]$.

7. **Identity matrix :** A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. Thus, the square matrix $A = [a_{ij}]$ is an identity matrix, if

$a_{ij} = \begin{cases} 1, \; if \; i = j \\ 0, \; if \; i \neq j \end{cases}$. We denote the identity matrix of order *n* by I_n.

Example : $[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2, and 3 respectively.

8. **Scalar matrix :** A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix. Thus, if $A = [a_{ij}]$ is a square matrix and $a_{ij} = \begin{cases} \alpha, \; if \; i = j \\ 0, \; if \; i \neq j \end{cases}$, then *A* is a scalar matrix.

Unit matrix and null square matrices are also scalar matrices

e.g. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ are scalar matrices

9. **Triangular matrix :** A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types.

(i) **Upper triangular matrix :** A square matrix $[a_{ij}]$ is called the upper triangular matrix if $a_{ij} = 0$ when $i > j$.

Example : $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix of order 3×3.

(ii) **Lower triangular matrix :** A square matrix $\left[a_{ij}\right]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

Example : $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3×3.

A triangular matrix is said to be strictly triangular if $a_{ii} = 0$ for $1 \le i \le n$.

ELEMENTARY TRANSFORMATION OF ELEMENTARY OPERATIONS OF A MATRIX

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

1. Interchange of any two rows (columns)

   It ith row (column) of a matrix is interchanged with the jth row (column), it will denoted by

   $R_i \leftrightarrow R_j \left(C_i \leftrightarrow C_j\right)$.

   Example : $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$, we get $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

2. Multiplying all elements of a row (column) of a matrix by a non-zero scalar.

   If the elements of ith row (column) are multiplied by non-zero scalar k, it will be denoted by

   $R_i \to R_i(k)\left[C_i \to C_i(k)\right]$ or $R_i \to k\,R_i\left[C_i \to k\,C_i\right]$.

   Example : $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_2 \to 3R_2$, we obtain $B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 6 \\ -1 & 2 & -3 \end{bmatrix}$

3. Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar *k*.

   If *k* times the elements of jth row (column) are added to the corresponding elements of the ith row (column), it will be denoted by $R_i \to R_i + k\,R_j \left(C_i \to C_i + k\,C_j\right)$.

TRACE OF A MATRIX

The sum of diagonal elements of a square matrix A is called the trace of matrix A, which is denoted by *tr* A.

$$tr\ A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots a_{nn}$$

MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB if the number of columns in *A* (pre-multiplier) is same as the number of rows in B (post multiplier). Thus if $A = \left[a_{ir}\right]_{m \times n}$ *and* $B = \left[b_{rj}\right]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as $AB = \left[C_{ij}\right]_{m \times p}$ where $\left(C_{ij}\right) = \sum_{r=1}^{n} a_{ir} b_{rj}$.

Now we define the product of a row matrix and a column matrix.

Let $A=\left[a_1 b_1 ... a_n\right]$ be a row matrix and $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column matrix.

Then $AB=\left[a_1 b_1 + a_2 b_2 + ... + a_n b_n\right]$

PROPERTIES OF MATRIX MULTIPLICATION

If A . B. and C are three matrices such that their product is defined, then

1. $AB \neq BA$ (Generally not commutative)
2. $(AB)C = A(BC)$ (Associative Law)
3. $IA = A = AI$, where I is identity matrix for multiplication.
4. $A(B+C) = AB + AC$, (Distributive Law)
5. If $AB = AC \not\Rightarrow B = C$, (Cancellation law is not applicable)
6. If $AB = 0$ it does not mean that A= 0 or B = 0, again product of two non zero matrix may be a zero matrix.
7. If $AB = BA$ then matrices A and B are called **commutative** matrices
8. If $AB = -BA$ then A and B are called **anti-commutative** matrices.

POSITIVE INTEGRAL POWERS OF A MATRIX

The positive integral power of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A.A, A^3 = A.A.A = A^2 A.$ also for any positive integers *m* and *n*,

(i) $A^m A^n = A^{m+n}$ (ii) $\left(A^m\right)^n = A^{mn} = \left(A^n\right)^m$

(iii) $I^n = I$, (iv) $A^0 = I_n$, where A is a square matrix of order n.

TRANSPOSE OF MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A denoted by A^T or A'

From the definition it is obvious that if order of A is $m \times n$, then order of A^T is $n \times m$.

Example : Transpose of matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{2\times3}$ is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}_{3\times2}$

PROPERTIES OF TRANSPOSE

Let A and B be two matrices then

1. $\left(A^T\right)^T = A$
2. $(A+B)^T = A^T + B^T$, A and B being of the same order
3. $(kA)^T = kA^T$, k be any scalar (real or complex)
4. $(AB)^T = B^T A^T$, A and B being conformable for the product AB
5. $\left(A_1 A_2 A_3 .... A_{n-1} A_n\right)^T = A_n^T A_{n-1}^T ...... A_3^T A_2^T A_1^T$
6. $I^T = I$

SPECIAL TYPES OF MATRICES

1. **Symmetric matrix:** In square matrix if $a_{ij} = a_{ji}$ for all i, j *or* $A^T = A$ is called symmetric matrix.

Example : $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

2. **Skew –symmetric matrix :** A square matrix A, $A=\left[a_{ij}\right]$ is called skew-symmetric matrix if $a_{ij}=-a_{ji}$ for all i, j or $A^T=-A.$

Example : $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

All principle diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element.

$$a_{ii}=-a_{ii} \quad \Rightarrow \quad a_{ii}=0$$

PROPERTIES OF SYMMETRIC AND SKEW -SYMMETRIC MATRICES

1. If A is a square matrix, then $A+A^T, AA^T, A^TA$ are symmetric, while $A-A^T$ is skew – symmetric matrix.
2. If A is symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^T\ AB$ are symmetric matrices where $n\in N$, $K\in R$ and B is a square matrix of order that of A.
3. If A is a skew – symmetric matrix, then
   (i) A^{2n} is a symmetric matrix for $n\in N$
   (ii) A^{2n+1} is a skew – symmetric matrix for $n\in N$
   (iii) kA is also skew – symmetric matrix, where $k\in R.$.
   (iv) $B^T\ AB$ is also skew-symmetric matrix where B is a square matrix order that of A.
4. If A, B are two symmetric matrices, then
   (i) $A\pm B$, $AB+BA$ are also symmetric matrices,
   (ii) $AB-BA$ is a skew – symmetric matrix,
   (iii) AB is also skew – symmetric when $AB=BA$.
5. If A, B are two skew – symmetric matrices, then
   (i) $A\pm B$, $AB-BA,$ are skew symmetric matrices, then
   (ii) AB + BA is a symmetric matrix.
6. If A is a skew – symmetric matrix and C is a column matrix, then $C^T\ AC$ is a zero matrix.
7. Every square matrix A can uniquely be expressed as sum of a symmetric and skew-symmetric matrix
   i.e., $A=\left[\frac{1}{2}\left(A+A^T\right)\right]+\left[\frac{1}{2}\left(A-A^T\right)\right].$

SINGULAR AND NON SINGULAR MATRIX

Any square matrix A is said to be singular if $|A|=0,$ and a square matrix A is said to be non singular if $|A|\neq 0,$ Here $|A|$ means corresponding determinant of square matrix A.

Example : $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ then $|A|=\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}=10-12=-2 \Rightarrow A$ is a non-singular matrix.

HERMITIAN AND SKEW-HERMITIAN MATRIX

A square matrix $A=\left[A_{ij}\right]$ is said to be **Hermitian Matrix**

if $a_{ij}=\overline{a}_{ji}; \forall i, j$ *i.e.*, $A=\left(\overline{A}\right)^T$.

Example : $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$, $\begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$

If A is a Hermitian matrix then $a_{ii} = \overline{a}_{ii} \Rightarrow a_{ii}$ is real $\forall i$, thus every diagonal element of a Hermitian matrix must be real.

A square matrix, $A = |a_{ij}|$ is said to be **Skew – Hermitian** if $a_{ji} = -\overline{a}_{ij}.\forall i,\ j\, i.e. A^T = -\overline{A}.$ If A is skew-Hermitian matrix, then $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii}$ must be purely imaginary or zero.

Example : $\begin{bmatrix} 0 & -2+i \\ 2-i & 0 \end{bmatrix}$, $\begin{bmatrix} 3i & -3+2i & -1-i \\ 3+2i & -2i & -2-4i \\ 1-i & 2-4i & 0 \end{bmatrix}$

ORTHOGONAL MATRIX

A square matrix A is called orthogonal if $AA^T = I = A^T A$ e.i.if $A^{-1} = A^T$

Example : $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal because $A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A^T$

In fact every unit matrix is orthogonal. Determinant of orthogonal matrix is –1 or 1

IDEMPOTENT MATRIX

A square matrix A is called an idempotent matrix if $A^2 = A$

Example : $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an idempotent matrix, because $A^2 = \begin{bmatrix} 1/4 + 1/4 & 1/4 + 1/4 \\ 1/4 + 1/4 & 1/4 + 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A$

Also $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotent matrices because $A^2 = A$ or $B^2 = B$

In fact every unit matrix is idempotent

INVOLUTORY MATRIX

A square matrix A is called an involutory matrix if $A^2 = I$ *or* $A^{-1} = A$

Example : $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an involutory matrix because $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

In fact every unit matrix is involutory.

NILPOTENT MATRIX

A square matrix A is called a nilpotent matrix if there exists a $p \in N$ such that $A^p = 0$.

Example : $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a nilpotent matrix because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ (Here P = 2)

Determinant of every nilpotent matrix is 0.

UNITARY MATRIX

A square matrix is said to unitary, if

$\overline{A}'A = I$ since $|\overline{A}'| = |A|$ *and* $|\overline{A}'A| = |\overline{A}'||A|$ therefore if $\overline{A}'A = I$ we have $|\overline{A}'||A| = 1.$

Thus he determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be non-singular.

Hence $\overline{A}'A = I \quad \Rightarrow \quad A\overline{A}' = I$

PERIODIC MATRIX

A matrix A will be called a periodic matrix if there exists a positive integer k such that $A^{k+1} = A$. If however k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A.

DIFFERENTIATION OF A MATRIX

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$ *then* $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A.

Example : If $A = \begin{bmatrix} x^2 & \sin x \\ 2x & 2 \end{bmatrix}$ then $\frac{dA}{dx} = \begin{bmatrix} 2x & \cos x \\ 2 & 0 \end{bmatrix}$

CONJUGATE OF A MATRIX

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$

Example : $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

PROPERTIES OF CONJUGATES

1. $\overline{\left(\overline{A}\right)} = A$
2. $\overline{(A+B)} = \overline{A} + \overline{B}$
3. $\overline{(\alpha A)} = \overline{\alpha}\,\overline{A}, \alpha$ being any number
4. $\overline{(AB)} = \overline{A}\,\overline{B}, A$ and B being conformable for multiplication.

TRANSPOSE CONJUGATE OF A MATRIX

The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A

i.e. $\overline{(A')} = \left(\overline{A}\right)' = A^\theta$

If $A = \left[a_{ij}\right]_{m\times n}$ then $A^\theta = \left[b_{ji}\right]_{n\times m}$ where $b_{ji} = \overline{a_{ij}}$

i.e. the $(j,i)^{th}$ elements of $A^\theta =$ the conjugate of $(i,\ j)^{\text{th}}$ element of A.

Example : If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then $A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

PROPERTIES OF TRANSPOSE CONJUGATE

1. $\left(A^\theta\right)^\theta = A$
2. $(A+B)^\theta = A^\theta + B^\theta$
3. $(kA)^\theta = \overline{k}\ A^\theta$, k being any number
4. $(AB)^\theta = B^\theta A^\theta$.

ADJOINT OF A SQUARE MATRIX

Let $A = \left[a_{ij}\right]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by $adj\ A$. Thus, $adj\ A = \left[C_{ij}\right]^T$

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T=\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

where C_{ij} denotes the cofactor of a_{ij} in A.

PROPERTIES OF ADJOINT MATRIX

If A, B are square matrices of order *n* and I_n is corresponding unit matrix, then

(i) $A(adjA)=|A|I_n=(adj\ A)A$ (Thus A (adj A) is always a scalar matrix)

(ii) $|adj\ A|=|A|^{n-1}$

(iii) $adj(adj\ A)=|A|^{n-2}A$

(iv) $|adj(adj\ A)|=|A|^{(n-1)^2}$

(v) $adj(A^T)=(adj\ A)^T$

(vi) $adj(AB)=(adj\ B)(adj\ A)$

(vii) $adj(A^m)=(adj\ A)^m, m\in N$

(viii) $adj(kA)=k^{n-1}(adj\ A), k\in R$

(ix) $adj(I_n)=I_n$

(x) $adj(\mathrm{O})=\mathrm{O}$

(xi) A is symmetric $\Rightarrow$ $adj\ A$ is also symmetric

(xii) A is diagonal $\Rightarrow$ $adj\ A$ is also diagonal.

(xiii) A is triangular $\Rightarrow$ $adj\ A$ is also triangular.

(xiv) A is singular $\Rightarrow$ $|adj\ A|=0$

Note : The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of off-diagonal (left hand side lower corner to right hand side upper corner) elements.

INVERSE OF A MATRIX

A non -singular square matrix of order *n* is invertible if there exists a square matrix *B* of the same order such that $AB=I_n=BA$.

In such a case, we say that the inverse of A is B and we write $A^{-1}=B$. The inverse of A is given $A^{-1}=\frac{1}{|A|}adj\ A$. The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A|\neq 0$.

PROPERTIES OF INVERSE MATRIX

If *A* and *B* are invertible matrices of the same order, then

1. Every invertible matrix possesses a unique inverse.
2. $(A^{-1})^{-1}=A$
3. $(A^T)^{-1}=(A^{-1})^T$
4. (Reversal Law) If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$

   In general, if $A, B, C,\ldots$ are invertible matrices then

   $$(ABC\ldots)^{-1}=\ldots C^{-1}B^{-1}A^{-1}.$$
5. $(A^k)^{-1}=(A^{-1})^k, k\in N\left[\text{In particular}(A^2)^{-1}=(A^{-1})^2\right]$
6. $adj(A^{-1})=(adj\ A)^{-1}$

7. $\left|A^{-1}\right| = \dfrac{1}{|A|} = |A|^{-1}$

8. $A = \text{diag}\ (a_1 a_2 ... a_n) \ \Rightarrow \ A^{-1} = diag\left(a_1^{-1}\, a_2^{-1} ... a_n^{-1}\right)$

9. A is symmetric $\Rightarrow$ A^{-1} is also symmetric

10. A is diagonal, $|A| \neq 0 \ \Rightarrow \ A^{-1}$ is also diagonal.

11. A is a scalar matrix $\Rightarrow$ A^{-1} is also a scalar matrix.

12. A is triangular, $|A| \neq 0 \ \Rightarrow \ A^{-1}$ is also triangular.

MINOR OF A MATRIX

Definition : Let A be a $m \times n$ matrix. If we retain any r rows and columns of *A* we shall have a square sub-matrix of order *r*. The determinant of the square sub-matrix of order *r* is called a minor of *A* order *r*. Consider any matrix A which is of the order of 3×4 say, $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 8 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$. It is 3×4 matrix so we can have minor, of order 3, 2 or 1 . Taking any three rows and three columns we get a minor of order three. Hence a minor of order $3 = \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 8 \\ 1 & 5 & 0 \end{vmatrix}$

Similarly we can consider any other minor of order 3. Minor of order 2 is obtained by taking any two rows and any two columns.

Minor of order $2 = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$ Minor of order 1 is every matrix of any element of the matrix.

RANK OF A MATRIX

The rank of a given matrix A is said to be *r* if

1. Every minor of *A* of order $r+1$ is zero.
2. There is at least one minor of *A* of order *r* which does not vanish. Here we can also say that the rank of a matrix *A* is *r*, if
   (i) Every square sub matrix of order $r + 1$ is singular.
   (ii) There is at least one square sub matrix of order *r* which is non-singular.

   The rank *r* of matrix *A* is written as $p(A) = r$ Echelon form of a matrix.

A matrix *A* is said to be in Echelon form if either *A* is the null matrix or *A* satisfies the following conditions :

1. Every non- zero row in *A* precedes every zero row.
2. The number of zeros before the first non zero element in a row is less than the number of such zeros in the next row.

   Rank of a matrix in Echelon form: The rank of a matrix in Echelon form is equal to the number of non zero rows in that matrix.

   Homogeneous and non- homogeneous systems of linear equations : A system of equations $AX = B$ is called a homogeneous system if $B = O$. If $B \neq O$, it is called a non-homogeneous system of equations.

   e.g., $2x + 5y = 0$
   $3x - 2y = 0$

   is a homogeneous system of linear equations whereas the system of equations given by

   e.g., $2x + 3y = 5$
   $x + y = 2$

is a non-homogeneous system of linear equations.

SOLUTION OF NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

(i) **Matrix method :**

(a) If A is non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.

(b) If A is singular matrix, and (adj A) $D = 0$, then the system of equations given by $AX = D$ is consistent with infinitely many solutions.

(c) If A is singular matrix and (adj A) $D \neq 0$, then the system of equation given by $AX = D$ is inconsistent.

(ii) Rank method for solution of Non-Homogeneous system $AX = B$.

(a) Write down A, B

(b) Write the augmented matrix $[A : B]$

(c) Reduce the augmented matrix to Echelon form by using elementary row operations.

(d) Find the number of non-zero rows in A and $[A : B]$ to find the ranks of A and $[A : B]$ respectively.

(e) If $\rho(A) \neq \rho(A : B)$, then the system is inconsistent.

(f) $\rho(A) = \rho(A : B) =$ the number of unknowns, then the system has a unique solution.

If $\rho(A) = \rho(A : B) <$ number of unknowns, then the system has an infinite number of solutions.

SOLUTIONS OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Let $AX = O$ be a homogeneous system of 3 linear equations in 3 unknowns.

(a) Write the given system of equations in the form $AX = O$ and write A.

(b) Find $|A|$.

(c) If $|A| \neq 0$, then the system is consistent and $x = y = z = 0$ is the unique solution.

(d) If $|A| = 0$, then the systems of equations has infinitely many solutions. In order to find that put $z = K$ (any real number) and solve any two equations for x and y so obtained with $z = K$ give a solution of the given system of equations.

Consistency of a system of linear equation *AX* = *B*, where *A* is a square matrix

In system of linear equations $AX = B$, $A = (a_{ij})_{n\times n}$ is said to be

(i) Consistent (with unique solution) if $|A| \neq 0$. i.e., if A is non-singular matrix.

(ii) Inconsistent (it has no solution) if $|A| = 0$ and $(adj\ A)B$ is a non-null matrix.

(iii) Consistent (with infinitely m any solutions) if $|A| = 0$ and $(adj\ A)B$ is a null matrix.

Cayley-Hamilton theorem

Every square matrix satisfies its characteristic equation e.g. let A be a square matrix then $|A - xI| = 0$ is the characteristics equation of A. If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristics equation for A, then

$$A^3 - 4A^2 + 5A - 7I = 0.$$

Roots of characteristic equation for A are called Eigen values of A or characteristic roots of A or latent roots of A.

If λ is characteristic root of A, then λ^{-1} is characteristic root of A^{-1}.

Sequence & Series Chapter 7

SEQUENCE

A sequence is a function whose domain is the set N of natural numbers and range a subset of real numbers or complex numbers.

A sequence whose range is a subset of real numbers is called a real sequence. Since we shall be dealing with real sequences only, we shall use the term sequence to denote a real sequence.

NOTATION

The different terms of a sequence are usually denoted by $a_1, a_2, a_3, ...$ or by $t_1,\ t_2,\ t_3,\ ....$The subscript (always a natural number) denotes the position of the term in the sequence. The number occurring at the *n*th place of a sequence is called the general term of the sequence.

Note :

A sequence is said to be finite or infinite according as it has finite or infinite number of terms.

SERIES

By adding or subtracting the terms of a sequence, we obtain a series.

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

PROGRESSIONS

If the terms of a sequence follow certain pattern, then the sequence is called a progression. Three special types of progressions are :-

(i) Arithmetic Progression (A.P.)
(ii) Geometric Progression (G.P.)
(iii) Harmonic Progression (H.P.)

ARITHEMTIC PROGRESSION

Arithmetic Progression or Arithmetic Sequence: A succession of number is said to be in Arithmetic progression (A.P) if the difference between any term and the term preceding it is constant throughout. This constant is called the common different (c.d.) of A.P.

In an A.P., we usually denote the first term by a, the common difference by d and the nth term by t_n.

Clearly, $d = t_n - t_{n-1}$.

Thus, an A.P. can be written as $a, a+d, a+2d, ..., a+(n-1)d, ...$

THE *n* TH TERM OF AN ARITHMETIC PROGRESSION

If a is the first term and d is the common difference of an A.P., then its nth term t_n is given by

$$t_n = a + (n-1)d$$

Note :

(a) If an A.P. has n terms, then the nth term is called the last term of A.P. and it is denoted by.
$\therefore \quad l = a + (n-1)d.$

(b) Three numbers a, b, c, are in A.P. if and only if $b - a = c - b$ or if $a + c$ =2b.

(c) If a is the first term and d the common difference of an A.P. having m terms, then nth term from the end is $(m-n+1)$th term from the beginning $\therefore$ *n th* term from the end $= a + (m-n)d$.

(d) Any three numbers in an A.P. can be taken as $a-d, a, a+d$.
Any four numbers in an A.P. can be taken as $a-3d, a-d, a+d, a+3d$.
Similarly 5 numbers in A.P. can be taken as $a-2d, a-d, a, a+d, a+2d$.

SUM OF *n* TERMS OF AN A.P.

The sum of n terms of an A.P. with first term 'a' and common difference '*d*' is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Note :

(a) If S_n be the sum of n terms of an A.P. whose first term is a and last term is l then $S_n = \frac{n}{2}(a+l)$.

(b) If we are given the common difference d, number of terms n and the last term l then

$$S_n = \frac{n}{2}\left[2l - (n-1)d\right]$$

(c) $t_n = S_n - S_{n-1}$.

PROPERTIES OF A.P.

I. If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

1. $a_1 + k, a_2 + k, ..., a_n + k$ are also in A.P.
2. $a_1 - k, a_2 - k, ..., a_n - k$ are also in A.P.
3. $ka_1, ka_2, ..., ka_n$ are also in A.P.
4. $\frac{a_1}{k}, \frac{a_2}{k}, ..., \frac{a_n}{k}, k \neq 0$ are also in A.P.

II. If $a_1, a_2, a_3, ...$ and b_1, b_2, b_3, ... are two A.P.s, then

1. $a_1 + b_1, a_2 + b_2, a_3 + b_3, ...$ are also in A.P.
2. $a_1 - b_1, a_2 - b_2, a_3 - b_3, ...$ are also in A.P.

III. If $a_1, a_2, a_3, ..., a_n$ are in A.P., then

1. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = ...$
2. $a_r = \frac{a_{r-k} + a_{r+k}}{2}, 0 \le k \le n - r$.
3. If number of terms of any A.P is odd, then sum of the terms is equal to product of middle term and number of terms.
4. If number of terms of any A.P. is odd then its middle terms is A.M. of first and last term.

ARITHEMETIC MEAN (A.M.)

SINGLE ARITHMETIC MEAN

A number A is said to be the single A.M. between two given numbers a and b if a, A, b are in A.P.

***n* ARITHMETIC MEANS**

The numbers $A_1, A_2, ..., A_n$ are said to be the n arithmetic means between two given numbers a and b if $a, A_1, A_2, ..., A_n$, b are in A.P. For example, since 2,4,6,8,10,12 are in A.P., therefore 4,6,8,10 are the four arithmetic means between 2 and 12.

INSERTING SINGLE A.M. BETWEEN TWO GIVEN NUMBERS

Let *a* and *b* be two given numbers and A be the A.M. between them. Then, *a* ,*A*, *b* are in A.P.

$$\therefore \quad A - a = b - A \text{ or } 2A = a + b, \quad \therefore \quad A = \frac{a+b}{2}.$$

INSERTINS *n*-ARITHMETIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $A_1, A_2, ..., A_n$ be the n arithmetic means between two given numbers *a* and *b*.

Then $a, A_1, A_2, ...A_n, b$ are in A.P. Now, $b = (n+2)th$ term of A.P.

$b = a + (n+1)d \;\; \therefore d = \frac{b-a}{n+1}$, where d is common difference of A.P.

$$\therefore A_1 = a + d = a + \left(\frac{b-a}{n+1}\right), A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right),\ A_n = a + nd = a + n\left(\frac{b-a}{n+1}\right)$$

Note :

The sum of n arithmetic means between two given numbers is n times the single A.M. between them i.e. if a and b are two given numbers and $A_1, A_2, ..., A_n$ are n arithmetic means between them, then

$$A_1 + A_2 + ... + A_n = n\left(\frac{a+b}{2}\right).$$

GEOMETRIC PROGRESSION

A succession of numbers is said to be in .G.P if the ratio of any term and the term preceding it is constant throughout. This constant is called the common ratio of the .G.P.

The constant ratio is usually denoted by r and is called the common ratio of the G.P.

***n* TH TERM OF A G.P.**

If a is the first and r is the common ratio of a G. P., then its nth term t_n is given by $t_n = ar^{n-1}$.

Note :

(a) If a is the first term and r is the common ratio of a G.P. then the G.P. can be written as

$$a, ar, ar^2, ..., ar^{n-1}, ...(a \neq 0).$$

(b) If a is the first term and r is the common ratio of a finite G.P. consisting of m terms, then the *n*th term from the end is given by ar^{m-n}.

(c) The nth term from the end of a G.P. with last term l and common ratio r is $\frac{l}{r^{(n-1)}}$.

(d) Three numbers in G.P. can be taken as $\frac{a}{r}, a, ar$

Four numbers in G.P. can be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

Five numbers in G.P. can be taken as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

(e) Three numbers $a, b, c,$ are in G.P. if and only if $\frac{b}{a} = \frac{c}{b}$ or if $b^2 = ac$.

SUM OF *n* TERMS OF A G.P.

The sum of first n terms of a G.P. with first term 'a' and common ratio 'r' is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}, r \neq 1.$$

Note :

(a) When $r = 1,\ S_n = a + a + a + ...$ upto n terms $= na$.

(b) If l is the last term of the G.P., then $S_n = \frac{lr - a}{r-1}, r \neq 1.$

SUM OF AN INFINITE G.P.

The sum of an infinite G.P. with first term 'a' and common ratio is 'r', when

(i) when $|r| < 1$ then $S_\infty = \frac{a}{1-r}$ (ii) when $|r| > 1$ then $S_\infty = \infty$

PROPERTIES OF G.P.

I. If $a_1, a_2, a_3, ...$ are in G. P., then

1. $a_1k, a_2k, a_3k, ...$ are also in G.P.

2. $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, ...$ are also in G.P.
3. $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ...$ are also in G.P.

II. If $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ are two G.P. s, then,

1. $a_1b_1, a_2b_2, a_3b_3, ...$ are also in G.P.
2. $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, ...$ are also in G.P.

III. 1. In a finite G.P. the product of terms equidistant from the beginning and the end is always the same and is equal to the product of the first and last term. If $a_1,\ a_2,\ a_3,\ ...,\ a_n$ be in G.P

Then, $a_1a_n = a_2a_{n-1} = a_3a_{n-3} = \ ... \ = a_r \cdot a_{n-r+1}$

2. $a_r = \sqrt{a_{r-k}\, a_{r+k}}\ ,\quad 0 \le k \le n-r.$
3. If the terms of a given G.P. are chosen at regular intervals. Then the new sequence so formed also forms a G.P.
4. If $a_1, a_2, a_3 .... a_n ...$ is a G.P. of non-zero, non-negative terms, then $\log a_1, \log a_2, \log a_3, .. \log a_n, ...$ is an A.P and vice-versa.
5. If first term of a G.P. of n terms is a and last term is $l,$ then the product of all terms of the G.P. is $(al)^{n/2}$
6. If there be n quantities in G.P. whose common ratio is r and S_m denotes the sum of the first m terms, then the sum of their product taken two by two is $\frac{r}{r+1} S_n S_{n-1}.$
7. If $a^{x_1}, a^{x_2}, a^{x_3}, ..., a^{x_n}$ are in G.P. then $x_1, x_2, x_3, ... x_n$ will be in A.P.

GEOMETRIC MEAN (G.M) : SINGLE GEOMETRIC MEAN

A number G is said to be the single geometric mean between two given numbers *a* and *b* if a, G, b are in G.P.

For example, since $2, 4, 8$ are in G.P., therefore 4 is the G.M. between 2 and 8 .

***n*-GEOMETRIC MEANS**

The numbers $G_1, G_2, ..., G_n$ are said to be the n geometric means between two given positive numbers a and b if $a, G_1, G_2, ...,\ G_n$, b are in G.P.

For example, since $1, 2, 4, 8, 16$ are in G.P., therefore 2,4, 8 are the three geometric means between 1 and 16 .

INSERTING SINGLE G.M. BETWEEN TWO GIVEN NUMBERS

Let a and b be two given positive numbers and G be the G.M. between them. Then a, G, b are in G.P.

$$\therefore \quad \frac{G}{a} = \frac{b}{G} \quad \text{or} \quad G^2 = ab, \quad \therefore \quad G = \sqrt{ab}$$

INSERTING n GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $G_1, G_2, G_3, ..., G_n,$ be the n geometric means between two given numbers a and b.

Then, $a, G_1, G_2, G_3, ..., G_n, b$ are in G.P. Now, $b = (n+2)$ th term of G.P. $= ar^{n+1},$ where r is the common ratio

$$\therefore r^{n+1} = \frac{b}{a} or \ r = \left(\frac{b}{a}\right)^{1/n+1} \quad \therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/n+1},\ G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/n+1},\ ...,\ G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}$$

Note :

1. The product of n geometric means between two given numbers is nth power of the single G.M. between them i.e. if a and b are two given numbers and $G_1, G_2, ..., G_n$ are n geometric means between them, then $G_1 G_2 G_3 \ ... \ G_n = \left(\sqrt{ab}\right)^n$.
2. If A and G are respectively arithmetic and geometric means between two positive numbers a and b then
(a) $A > G$
(b) The quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$
(c) the two positive numbers are $A \pm \sqrt{A^2 - G^2}$.

HARMONIC PROGRESSION

A sequence of non –zero numbers $a_1, a_2, a_3, ...$ is said to be a harmonic progression if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ...$ is an A.P.

***n* TH TERM OF AN H.P.**

nth term of H.P. $= \dfrac{1}{\text{nth term of the corresponding A.P.}}$

HARMONIC MEAN (H.M)

SINGLE HARMONIC MEAN

A number H is said to be the single harmonic mean between two given numbers a and b if a, H, b are in H.P. For example, since $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are in H.P., therefore, $\frac{1}{3}$ is the H.M. between $\frac{1}{2}$ and $\frac{1}{4}$

***n*-HARMONHIC MEANS**

The number $H_1, H_2, ..., H_n$ are said to be the n harmonic means between two given numbers a and b if $a, \mathrm{H}_1, \mathrm{H}_2, ..., \mathrm{H}_n, b$ are in H.P. *i.e.* $\frac{1}{a}, \frac{1}{\mathrm{H}_1}, \frac{1}{\mathrm{H}_2}, ..., \frac{1}{\mathrm{H}_n}, \frac{1}{b}$ are in A.P.

INSERTING SINGLE H.M. BETWEEN TWO GIVEN NUMBERS

Let a and b be two given numbers and H be the H.M .between them .Then, a, H, b are in H.P.

$$\therefore H = \frac{2ab}{a+b}.$$

INSERTING n– HARMONIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $\mathrm{H}_1, \mathrm{H}_2, ... \mathrm{H}_n$ be the n harmonic means between two given numbers a and b.

Then, $a, \mathrm{H}_1, \mathrm{H}_2, ..., \mathrm{H}_n, b$ are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ ... \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

$d = \dfrac{a-b}{ab(n+1)}$. where d is common difference of the corresponding A.P.

$$H_1 = \frac{ab(n+1)}{bn+a}, \qquad H_2 = \frac{ab(n+1)}{2a+(n-1)b}, ..., \qquad H_n = \frac{ab(n+1)}{na+b}.$$

Note :

1. Three numbers a, b, c are in H.P. if and only if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. i.e. $\frac{1}{a} + \frac{1}{c} = 2.\frac{1}{b}$ $i.e. b = \frac{2ac}{a+c}$.
2. No term of H.P. can be zero

3. There is no general formula for finding the sum of n terms of H.P.
4. Reciprocals of terms of H.P. are in A.P. and then properties of A.P. can be used.
5. If H is the H.M. between *a* and *b*, then

(i) $\frac{1}{H-a}+\frac{1}{H-b}=\frac{1}{a}+\frac{1}{b}$

(ii) $(H-2a)(H-2b)=H^2$

(iii) $\frac{H+a}{H-a}+\frac{H+b}{H-b}=2$

RELATION BETWEEN A.M., G.M. AND H.M.

Let A,G, and H be arithmetic, geometric and harmonic means between two positive numbers a and b, then,

(i) $A=\frac{a+b}{2},\ G=\sqrt{ab}\ \ and\ H=\frac{2ab}{a+b}$

(ii) $G^2=AH$

(iii) $A\ge G\ge H$ equality will hold when $a=b$

RELATION BETWEEN A.P, G.P. AND H.P.

1. If A,G, H be A.M., G.M., and H.M between a and b, then $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\begin{cases} A & when\, n=0 \\ G & when\, n=-1/2 \\ H & when\, n=-1 \end{cases}$

2. If A_1, A_2 be two $A.M's$ G_1, G_2 be two G.M.'s and H_1, H_2 be two H.M;s between two numbers *a* and *b*, then, $\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$.
3. Recognization of A.P., G.P., H.P., : If a, b, c are thee successive terms of a sequence.

If $\frac{a-b}{b-c}=\frac{a}{a}$, then a, b c, are in A. P.

If $\frac{a-b}{b-c}=\frac{a}{b}$, then a, b, c are in G. P.

If $\frac{a-b}{b-c}=\frac{a}{c}$, then a, b, c are in H. P.

4. The sum of the fourth power of first n natural numbers

$$=\sum n^4=1^4+2^4+3^4+........n^4$$

$$=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

SOME SPECIAL SEQUENCES

1. The sum of first n natural numbers $=\sum n=1+2+3+...+n=\frac{n(n+1)}{2}$.
2. The sum of squares of first n natural numbers $=\sum n^2=1^2+2^2+3^2+....+n^2=\frac{n(n+1)(2n+1)}{6}$.

3. The sum of cubes of the first n natural numbers = $\sum n^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

General rule for finding the values of recurring decimal : Let X denote the figure which do not recur and assume they are l in number. Let Y denote recurring period of consisting of m figures. Let R denote the value of recurring decimal.

then $R = .XYYY\ ... \quad \therefore \quad 10^l R = X.YYY$ … (I)

and $10^{l+m}.R = XY.YYY$ … (II)

$\therefore$ subtracting (I) from (II) we get

$$R = \frac{XY - X}{10^{l+m} - 10^l}$$

ARITHMETICO– GEOMETRIC SEQUENCE (A.G.S)

A sequence is said to be an arithmetico geometric sequence if its each term is the product of the corresponding terms of an A.P. and a G.P. *i.e. if* $a_1, a_2, a_3, ...$ is an A.P. and $b_1, b_2, b_3, ...$ is a G.P. , then the sequence $a_1b_1, a_2b_2, a_3b_3, ...$ is an arithmetico – geometric sequence.

An arithmetico-geometric sequence is of the form $ab, (a+d)br, (a+2d)br^2, (a+3d)br^3, ...$, its sum to n terms is given by

$$S_n = \frac{ab}{1-r} + \frac{bdr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{\left[a+(n-1)d\right]br^n}{1-r}$$

If $|r| < 1,$ the sum of its infinite number of terms is given by

$$\lim_{n\to\infty} S_n = S_\infty = \frac{ab}{1-r} + \frac{brd}{(1-r)^2}$$

NOTE : If $r \ge 1$ then S_∞ does not exist

RECURRENCE RELATION

Suppose you are given a relation among three consecutive terms of a series as

$\alpha u_{n+2} + \beta u_{n+1} + \gamma u_n = 0$ (recurrence relation)

then let $u_n = \lambda^n$ to get

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0$$

Find λ_1 & λ_2. Hence the general n^{th} term u_n is expressible as $u_n = A\lambda_1^n + B\lambda_2^n$, where A & B are determined from the given sequence. For four consecutive terms, the recurrence relation

$$\alpha u_{n+3} + \beta u_{n+2} + \gamma u_{n+1} + \delta u_n = 0$$

given a cubic equation for λ.

$$\alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta = 0$$

Then n^{th} term can be written as

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n$$

NOTE : This will work if the substitution $a_n = \lambda^n$ makes the recurrence relation free from n.

METHOD FOR FINDING SUM

Method of Difference

This method is applicable for both sum of n terms and sum of infinite number terms.

First suppose that sum of the series is S, then multiply it by common ratio of the G.P. and subtract. In this way, we shall get a Geometric series whose sum can be easily obtained.

If the difference of the successive terms of a series in Arithmetic series or Geometric series we can find n^{th} term of series by the following steps :

Step I : Denote the n^{th} term by T_n and the sum of the series upto n terms by S_n

Step II : Rewrite the given series with each term shifted by one place to the right.

Step III : By subtracting the later series from the former, find T_n.

Step IV : From T_n, S_n can be found by appropriate summation.

Lagrange's identity

$$\left(x_1^2+x_2^2+x_3^2+\ldots+x_n^2\right)\left(y_1^2+y_2^2+y_3^2+\ldots\ldots+y_n\right)^2-\left(x_1y_1+x_2y_2+\ldots\ldots+x_ny_n\right)^n$$
$$=\left(x_1y_2-x_2y_1\right)^2+\left(x_1y_3-x_3y_1\right)^2+\left(x_1y_4-x_4y_1\right)^2+\ldots\ldots+\left(x_1y_n-x_ny_1\right)^2$$
$$+\left(x_2y_3-x_3y_2\right)^2+\left(x_2y_4-x_4y_2\right)^2+\ldots\ldots+\left(x_2y_n-x_ny_2\right)^2$$
$$+\left(x_3y_4-x_4y_3\right)^2+\ldots\ldots+\left(x_3y_n-x_ny_3\right)^2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad +\left(x_{n-1}y_n-x_ny_{n-1}\right)^2$$

Note : For a odd positive integer to be perfect square. It is necessary that it should be of the form $8k+1$.

Inequalities

Chapter 8

1. **ELEMENTARY RESULTS**

(i) If $a > b$ and $b > c$, then $a > c$. In general, If $a_1 > a_2,\ a_2 > a_3,\ \dots,\ a_{n-1} > a_n$, then $a_1 > a_n$.

(ii) If $a > b$, then $a \pm c > b \pm c$, for each $c \in \mathrm{R}$.

(iii) If $a > b$ and c > 0, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

(iv) If $a > b$ and c < 0, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

(v) If $a > b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

(vi) If $a_i > b_i > 0$ for $i = 1, 2, \dots\dots, n$, then $a_1 a_2 \dots\dots a_n > b_1 b_2 \dots . b_n$.

(vii) If $0 < a < 1$ and r is a positive real number, then $0 < a^r < 1 < \frac{1}{a^r}$.

(viii) If $a > 1$ and r is a positive real number, then $a^r > 1 > \frac{1}{a^r}$.

(ix) If $0 < a < b$ and r is positive real number, then $a^r < b^r$ and $\frac{1}{a^r} > \frac{1}{b^r}$.

Also, $0 < \frac{a}{b} < 1, 0 < \left(\frac{a}{b}\right)^r < 1 < \left(\frac{b}{a}\right)^r$.

(x) Triangle Inequality $|a+b| \le |a| + |b|, a, b \in \mathrm{R}$ More generally,

$|a_1 + a_2 + \dots . + a_n| \le |a_1| + |a_2| + \dots . + |a_n|$.

(xi) For a real number 'a', $|a| = \begin{cases} a, & \text{if } a \ge 0 \\ -a, & \text{if } a < 0 \end{cases}$

$\therefore \quad |a| = \max .\{a, -a\}$.

(xii) $-|a| \le a \le |a|, \forall a \in \mathrm{R}$.

(xiii) If $b \ge 0$, then $|x - a| \le b \Leftrightarrow a - b \le x \le a + b$.

(xiv) If $a > 1$ and $x > y > 0$, then $\log_a x > \log_a y$.

(xv) If $0 < a < 1$ and $x > y > 0$, then $\log_a x < \log_a y$.

2. For n positive numbers $a_1, a_2, \dots ., a_n$, $\mathrm{A} = \frac{1}{n}(a_1 + a_2 + \dots .. + a_n)$, $G = (a_1 a_2 \dots ., a_n)^{1/n}$

$H = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots . + \frac{1}{a_n}}$ are called the Arithmetic Mean, the Geometric Mean and the Harmonic Mean respectively. Note. $\mathbf{A} \ge \mathbf{G} \ge \mathbf{H}$ The equality holds only when all the numbers are equal.

Also, A, G and H lie between the least and the greatest of $a_1, a_2, \dots ., a_n$.

3. For n positive real number $a_1, a_2, \dots ., a_n$ and n positive real numbers $w_1, w_2, \dots ., w_n$

$A_w = \dfrac{w_1a_1 + ... + w_na_n}{w_1 + ... + w_n}$ and $G_w = \left(a_1^{w_1} ....a_n^{w_n}\right)^{\frac{1}{w_1+....+w_n}}$ are called weighted A.M and weighted G.M. of $a_1, a_2, ..., a_n$ with the respective weights $w_1, w_2, ..., w_n$.

Note. $A_w \geq G_w$ and $A_w = G_w$ iff $a_1 = a_2 = ... = a_n$.

4. **CAUCHY-SCHWARZ INEQUALITY**

For real number $a_1, a_2, ..., a_n$ and

$b_1, b_2, ..., b_n$ $\left(a_1b_1 + a_2b_2 + ... + a_nb_n\right)^2 \leq \left(a_1^2 + a_2^2 + ... + a_n^2\right)\left(b_1^2 + b_2^2 + ... + b_n^2\right)$

The equality holds iff $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = ... = \dfrac{a_n}{b_n}$.

5. **TCHEBYCHEF INEQUALITY**

For any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that $a_1 \leq a_2 \leq ..... \leq a_n$ and $b_1 \leq b_2 \leq ..... \leq b_n$,

$n\left(a_1b_1 + a_2b_2 + ... + a_nb_n\right) \geq \left(a_1 + a_2 + ... + a_n\right)\left(b_1 + b_2 + ... + b_n\right)$

The equality holds if either all a's or all b's are equal. Moreover,

$\dfrac{a_1b_1 + a_2b_2 + ... + a_nb_n}{n} \geq \dfrac{a_1 + a_2 + ... + a_n}{n} \times \dfrac{b_1 + b_2 + ...b_n}{n}$.

6. **WEIERSTRASS' INEQUALITY'**

If $a_1, a_2 \cdots, a_n$ are n positive real numbers, then for $n \geq 2$

$\left(1 + a_1\right)\left(1 + a_2\right)\cdots\left(1 + a_n\right) > 1 + a_1 + a_2 + \cdots + a_n$ and

if $a_1, a_2, \cdots a_n$ are n positive real numbers less then 1, then

$\left(1 - a_1\right)\left(1 - a_2\right)\cdots\left(1 - a_n\right) > 1 - a_1 - a_2 - \cdots - a_n$

Arithmetic Mean of *m*th Power :

Let $a_1,\ a_2, ...,\ a_n$ be n positive real numbers (not all equal) and let m be a real number, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} > \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m \text{ if } m \in R - [0,\ 1]$$

However if $m \in (0,\ 1)$, then $\dfrac{a_1^m + a_2^m + ... + a_n^m}{n} < \left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right)^m$

Obviously if $m \in \{0,\ 1\}$, then $\dfrac{a_1^m + a_2^m + ... + a_n^m}{n} = \left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right)^m$.

LOGARITHM

1. $\log_{a^\beta} n = \dfrac{1}{\beta}\log_a n$
2. $a^{\log_c b} = b^{\log_c a}, \left(a, b, c > 0 \text{ and } c \neq 1\right)$
3. Usually $\log a^n = n\log a$ but if n is even $\log a^n = n\log|a|$

LOGARITHMIC INEQUALITIES

1. If $a>1, p>1 \Rightarrow \log_a p>0$
2. If $0<a<1, p>1 \Rightarrow \log_a p<0$
3. If $a>1, 0<p<1 \Rightarrow \log_a p<0$
4. If $p>a>1 \Rightarrow \log_a p>1$
5. If $a>p>1 \Rightarrow 0<\log_a p<1$
6. If $0<a<p<1 \Rightarrow 0<\log_a p<1$
7. If $0<p<a<1 \Rightarrow \log_a p>1$
8. If $\log_m a>b \Rightarrow \begin{cases} a>m^b, & \text{if } m>1 \\ a<m^b, & \text{if } 0<m<1 \end{cases}$
9. If $\log_m a<b \Rightarrow \begin{cases} a<m^b, & \text{if } m>1 \\ a>m^b, & \text{if } 0<m<1 \end{cases}$
10. $\log_p a>\log_p b$

    $\Rightarrow$ $a \geq b$ if base p if positive and >1or $a \leq b$ if base p is positive and <1 i.e. 0< p< 1.

Permutation & Combination

Chapter 9

FACTORIAL NOTATION

The continued product of first n natural numbers is called n factorial or factorial n and is denoted by $\lfloor n$ or $n!$

Thus, $\lfloor n$ or $n! = 1 \cdot 2 \cdot 3 \cdot 4 .... (n-1) n$

$= n(n-1)(n-2)....3 \cdot 2 \cdot 1$ (in reverse order)

Note :

(a) When n is a negative integer or a function, $n!$ is not defined. Thus, $n!$ is defined only for positive integers.

(b) According to the above definition, $0!$ makes no sense. However we define $0! = 1.$

(c) $n! = n(n-1)!$

(d) $(2n)! = 2^n \cdot n! [1 \cdot 3 \cdot 5 \cdot 7 ..... (2n-1)].$

FUNDAMENTAL PRINCIPLE OF COUNTING

The Sum Rule

Suppose that A and B are two disjoint events (Mutually exclusive) that is , they never occur together, Further suppose that A occurs in m ways and B in n ways. Then A or B can occur in m + n ways. This rule can also be applied to more than two mutually exclusive events.

The product Rule

Suppose that an event X can be decomposed into two stages, A and B. Let stage A occur in m ways and suppose that these stages are unrelated, in the sense that stage B occurs in n ways regardless of the outcome of stage A. Then event X occurs in mn ways. This rule is applicable even if event X can be decomposed in more than two stages.

PERMUTATION

Each of the different arrangement which can be made by taking some or all of given number of things or objects at a time is called a Permutation.

Note : Permutation of things means arrangement of things. The word arrangement is used if order of things is taken into accounts. Thus, if order of different things changes, then their arrangement also changes.

NOTATIONS

Let r and n be positive integers such that $1 \le r \le n.$ Then, the number of permutations of n different things, taken r at a time, is denoted by the symbol ${}^nP_r$ or $P(n, r).$

COMBINATIONS

Definition

Each of the different groups or selections which can be formed by taking some or all of a number of objects, irrespective of their arrangements, is called a combination.

NOTATIONS

The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$.

Thus, $${}^nC_r = \frac{n!}{r!(n-r)!} \quad (0 \le r \le n) \quad = \frac{{}^nP_r}{r!} \quad = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3 \cdot 2 \cdot 1}$$

If $r > n,$ then ${}^nC_r = 0.$

Difference between a Permutation and Combination

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) Each combination corresponds to many permutations. For example, the six permutations *ABC, ACB, BCA, BAC, CBA* and *CAB* correspond to the same combination *ABC.*

Number of Combinations without repetition

The number of combinations (selections or groups) that can be formed from n different objects taken $r\left(0 \le r \le n\right)$ at a time is ${}^{n}C_{r} = \dfrac{n!}{r!\left(n-r\right)!}$.

SOME USEFUL RESULTS OF PERMUTATIONS

1. $${}^{n}P_{r} = \frac{n!}{\left(n-r\right)!}$$
   $$= n\left(n-1\right)\left(n-2\right)...\left(n-r+1\right),\ 0 \le r \le n.$$
2. Number of Permutations of n different things taken all at a time ${}^{n}P_{n} = n!$
3. The number of permutations of n things, taken all at a time, out of which p are alike and are of one type, q are alike and are of second type and rest are all different $= \dfrac{n!}{p!q!}$.
4. The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^{r}.

Conditional Permutations

1. Number of permutations of *n* dissimilar things taken *r* at a time when *p* particular things always occur $= {}^{n-p}C_{r-p} \cdot r!$.
2. Number of permutations of *n* dissimilar things taken *r* at a time when *p* particular things never occur $= {}^{n-p}C_{r} r!$.
3. The total number of permutations of *n* different things taken not more than *r* at a time, when each thing may be repeated any number of times, is $\dfrac{n\left(n^{r}-1\right)}{n-1}$.
4. Number of permutations of *n* different things, taken all at a time, when *m* specified things always come together is $m! \times \left(n-m+1\right)!$.
5. Number of permutations of *n* different things, taken all at a time, when *m* specified things never come together is $n! - m! \times \left(n-m+1\right)!$.
6. Let there be *n* objects, of which *m* objects are alike of one kind, and the remaining (*n* – *m*) objects are alike of another kind. Then, the total number of mutually distinguishable permutations that can be formed from these objects is $\dfrac{n!}{\left(m!\right) \times \left(n-m\right)!}$.

   The above theorem can be extended further i.e., if there are *n* objects, of which p_1 are alike of one kind; p_2 are alike of another kind; p_3 are alike of 3rd kind; ….; p_r are alike of rth kind such that $p_1 + p_2 + ... + p_n = n;$ then the number of permutations of these n objects is $\dfrac{n!}{\left(p_1!\right) \times \left(p_2!\right) \times .... \times \left(p_r!\right)}$.

Circular Permutations

In circular permutations, what really matters is the position of an object relative to the others.

Thus, in circular permutations, we fix the position of the one of the objects and then arrange the other objects in all possible ways.

There are two types of circular permutations :

(i) The circular permutations in which clockwise and the anticlockwise arrangements give rise to different permutations, e.g. seating arrangements of persons round a table.

(ii) The circular permutations in which clockwise and the anticlockwise arrangements give rise to same permutations, e.g. arranging some beads to form a necklace.

Difference between Clockwise and Anti-Clockwise Arrangement

If anti-clockwise and clockwise order of arrangement are not distinct e.g., arrangement of beads in a necklace, arrangement of flowers in garland etc. then the number of circular permutations of n distinct items is $\frac{(n-1)!}{2}$.

Note :

(a) The number of circular permutations of n different objects is $(n-1)!$.

(b) The number of ways in which n persons can be seated round a table is $(n-1)!$.

(c) The number of ways in which n different beads can be arranged to form a necklace, is $\frac{1}{2}(n-1)!$.

(d) Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are taken as different is $\frac{{}^nP_r}{r}$.

(e) Number of circular permutations of n different things, taken r at a time, when clockwise and anticlockwise orders are not different is $\frac{{}^np_r}{2r}$.

PROPERTIES OF ^{n}C$_r$

1. ${}^nC_r = {}^nC_{n-r}$
2. ${}^nC_0 = {}^nC_n = 1,\ {}^nC_1 = n$
3. If ${}^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n$.
4. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
5. $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$
6. $n \cdot {}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$
7. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
8. If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.
9. If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
10. ${}^nC_r = \frac{r \text{ decreasing numbers starting with } n}{r \text{ increasing numbers starting with } 1} = \frac{n(n-1)(n-2)...(n-r+1)}{1\cdot 2\cdot 3....r}$.
11. ${}^nP_r = r$ decreasing numbers starting with n $= n(n-1)(n-2)...(n-r+1)$.
12. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$.
13. ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ... = 2^{n-1}$.

Number of Combinations with repetition and all possible selections

(i) The number of combinations of n distinct objects taken r at a time when any object may be repeated any number of times. = Coefficient of x^r in $(1 + x + x^2 + ... + x^r)^n$

= Coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$

(ii) The total number of ways in which it is possible to form groups by taking some or all of n things at a time is ${}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$.

(iii) The total number of ways in which it is possible to make groups by taking some or all out of

$n = (n_1 + n_2 + ...)$ Things, when n_1 are alike of one kind, n_2 are alike of second kind, and so on is $\{(n_1+1)(n_2+1)....\}-1$.

(iv) The number of selections of r objects out of n identical objects is 1

(v) Total number of selections of zero or more objects from n identical objects is $n + 1$.

(vi) The number of selections taking at least one out of $a_1+a_2+a_3+....+a_n+k$ objects, where a_1 are alike (of one kind),a_2 are alike (of second kind) and ….a_n are alike (of n^{th} kind) and k are distinct

$$=\left[(a_1+1)(a_2+1)(a_3+1).....(a_n+1)\right]2^k-1.$$

Conditional Combinations

1. The number of ways in which r objects can be selected from n different objects if k particular objects are

(i) Always included = ${}^{n-k}C_{r-k}$ (ii) Never included = ${}^{n-k}C_r$

2. The number of combinations of n objects, of which p are identical, taken r at a time is

$${}^{n-p}C_r+{}^{n-p}C_{r-1}+{}^{n-p}C_{r-2}+....+{}^{n-p}C_0,\text{ if } r\le p \text{ and}$$

$${}^{n-p}C_r+{}^{n-p}C_{r-1}+{}^{n-p}C_{r-2}+....+{}^{n-p}C_{r-p},\text{ if } r>p.$$

Division into groups

Case I : 1. The number of ways in which n different things can be arranged into r different groups is ${}^{n+r-1}p_n$ or $n!\,{}^{n-1}C_{r-1}$ according as blank group are or not admissible.

2. The number of ways in which n different things can be distributed into r different groups is $r^n-{}^rC_1(r-1)^n+{}^rC_2(r-2)^n-....+(-1)^{r-1}\,{}^rC_{r-1}$ or Coefficient of x^n *is* $n!(e^x-1)^r$.

Here blanks groups are not allowed.

3. The number of distributions of and different things are difference parcel in which no one of x of the x assigned parcels is blank is $r^n-{}^xC_1(r-1)^n+{}^xC_2(r-2)^n........+(-1)^x(r-x)^n$

4. Number of ways in which $m\times n$ different objects can be distributed equally among n persons (or numbered groups) = (number of ways of dividing into groups) × (number of groups) ! $=\dfrac{(mn)!\,n!}{(m!)^n\,n!}=\dfrac{(mn)!}{(m!)^n}$.

5. If $N=a^pb^qc^r.....$ where a,b,c are distinct primes and $p,q,r...$ are any positive integers then number of all positive integers which are less than N and are prime to it are $N\left(1-\dfrac{1}{a}\right)\left(1-\dfrac{1}{b}\right)\left(1-\dfrac{1}{c}\right).....$ It is called Euler's Totient function.

Case II : The number of ways in which $(m+n)$ different things can be divided into two groups which contain m and n things respectively is, ${}^{m+n}C_m.{}^nC_n=\dfrac{(m+n)!}{m!\,n!},m\ne n.$

Corollary **:** If $m=n$, then the groups are equal size. Division of these groups can be given by two types.

Type I : If order of group is not important : The number of ways in which $2n$ different things can be divided equally into two groups is $\dfrac{(2n)!}{2!(n!)^2}$.

Type II : If order of group is important :

1. The number of ways in which $2n$ different things can be divided equally into two distinct groups is

$$=\frac{(2n)!}{2!(n!)^2}\times 2!=\frac{2n!}{(n!)^2}.$$

2. The number of ways in which (m + n + p) different things can be divided into three groups which contain m, n and p things respectively is ${}^{m+n+p}C_m.{}^{n+p}C_n.{}^{p}C_p=\frac{(m+n+p)!}{m!n!p!}, m\neq n\neq p.$

Corollary **:** If $m=n=p,$ then the groups are equal size. Division of these groups can be given by two types,

Type I : If order of group is not important : The number of ways in which 3p different thing can be divided equally into three groups is $=\frac{(3p)!}{3!(p!)^3}.$

Type II : If order of group is important : The number of ways in which 3p different things can be divided equally into three distinct groups is $=\frac{(3p)!}{3!(p!)^3}3!=\frac{(3p)!}{(p!)^3}.$

(i) If order of group is not important : The number of ways in which m n different things can be divided equally into n groups is $=\frac{mn!}{(m!)^n n!}.$

(ii) If order of group is important : The number of ways in which m n different things can be divided equally into n distinct groups is $=\frac{(mn)!}{(m!)^n n!}\times n!=\frac{(mn)!}{(m!)^n}.$

DERANGEMENT

Any change in the given order of the things is called a derangement. If n things form an arrangement in a row, the number of ways in which they can be deranged so that no one of them occupies its original place is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+....+(-1)^n.\frac{1}{n!}\right).$$

NUMBER OF RECTANGLES AND SQUARES

(a) Number of rectangles of any size in a square of size $n\times n$ is $=\sum_{r=1}^{n}r^3$

and number of squares of any size is $=\sum_{r=1}^{n}r^2.$

(b) Number of rectangles of any size in a rectangle of size $n\times p\,(n<p)$ is $=\frac{np}{4}(n+1)(p+1)$

and number of squares of any size is $=\sum_{r=1}^{n}(n+1-r)(p+1-r).$

EXPONENT OF PRIME p in $n!$

Exponent of a prime p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+...+\left[\frac{n}{p^k}\right].$$

where $p^k<n<p^{k+1}$ and $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}.$

For example, exponent of 3 in $(100)!$ is

$$E_3(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$$

SOME IMPORTANT RESULTS FOR GEOMETRICAL PROBLEMS

1. If n distinct points are given in the plane such that no three of which are collinear, then the number of line segments formed $= {}^nC_2$

   If m of these points are collinear $(m \geq 3)$, then the number of line segments is ${}^nC_2 - {}^mC_2 + 1$.
2. The number of diagonals in an n-sided closed polygon $= {}^nC_2 - n$.
3. If n distinct points are given in the plane such that no three of which are collinear, then the number of triangles formed $= {}^nC_3$.

   If m of these points are collinear $(m \geq 3)$, then the number of triangles formed $= {}^nC_3 - {}^mC_3$.
4. If n distinct points are given on the circumference of a circle, then
   (a) Number of st. lines $= {}^nC_2$
   (b) Number of triangles $= {}^nC_3$
   (c) Number of quadrilaterals $= {}^nC_4$ and so on
5. The sum of the digits in the unit place of all numbers formed with the help of non-zero digits $a_1, a_2, ..., a_n$ taken all at a time is $= (n-1)!(a_1 + a_2 + ... + a_n)$
   (repetition of digits not allowed)
6. The sum of all n digit numbers that can be formed using the digits $a_1, a_2, ..., a_n$ is

   $$= (n-1)!(a_1 + a_2 + ... + a_n)\frac{(10^n - 1)}{9}.$$ and if one of terms is zero digit then

   sum $= (n-1)!(a_1 + a_2 + ... + a_n)\left(\frac{10^n - 1}{a}\right) - (n-2)!(a_1 + a_2 + ... + a_n)\left(\frac{10^{n-1} - 1}{a}\right)$
7. $(n!)!$ is divisible by $(n!)^{(n-1)!}$
8. $(kn)!$ is divisible by $(n!)^k$
9. ${}^nC_r$ is divisible by n if n is prime $(n \neq r)$

Multinomial Theorem

Let $x_1, x_2, ..., x_m$ be integers. Then number of solutions to the equation

$$x_1 + x_2 + ... + x_m = n \qquad \text{… (i)}$$

Subject to the condition

$$a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, ..., a_m \leq x_m \leq b_m \qquad \text{… (ii)}$$

is equal to the coefficient of x^n in

$$\left(x^{a_1} + x^{a_1+1} + ... + x^{b_1}\right)\left(x^{a_2} + x^{a_2+1} + ... + x^{b_2}\right)....\left(x^{a_m} + x^{a_m+1} + ... + x^{b_m}\right) \qquad \text{… (iii)}$$

This is because the number of ways, in which sum of m integers in (i) equals n, is the same as the number of times x^n comes in the expansion of (iii).

Use of solution of linear and coefficient of a power in expansions to find the number of ways of distribution :

(i) The number of integral solutions of $x_1 + x_2 + x_3 + ... + x_r = n$ where $x_1 \geq 0, x_2 \geq 0, ...x_r \geq 0$ is the same as the number of ways to distribute *n* identical things among *r* persons.

This is also equal to the coefficient of x^n in the expansion of $\left(x^0 + x^1 + x^2 + x^3 + ...\right)^r$

$= \text{coefficient of } x^n \text{ in } \left(\frac{1}{1-x}\right)^r$

$= \text{coefficient of } x^n \text{ in } \left[(1-x)^{-r} = 1 + {}^rC_1 x + {}^{r+1}C_2 x^2 + {}^{r+2}C_3 x^3 + ...\right]$

$= {}^{r+n-1}C_n = {}^{n+r-1}C_{r-1}$

(ii) Similarly total number natural solutions of $x_1 + x_2 + ... + x_r = n$ are ${}^{n-1}C_{r-1}$

(iii) The number of possible arrangements permutations of P object out of n_1 identical objects of kind 1, n_2 identical objects of kind 2 and so on is

$= P! \times$ coefficient of x^P in the expansion of $\left(1 + x + \frac{x^2}{2!} + ... + \frac{x^{n_1}}{n_1!}\right) ..... \left(1 + x + \frac{x^2}{2!} + ... + \frac{x^{n_k}}{n_k!}\right)$

Number of Divisors

Let $N = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}....p_k^{\alpha_k}$, where $p_1, p_2, p_3, ...p_k$ are different prime and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are natural number then :

(i) The total number of divisors of N including 1 and N is = $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)....(\alpha_k + 1)$.

(ii) The total number of divisors of N excluding 1 and N is = $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)....(\alpha_k + 1) - 2$.

(iii) The sum of these divisors is =
$\left(p_1^0 + p_1^1 + p_1^2 + ... + p_1^{\alpha_1}\right)\left(p_2^0 + p_2^1 + p_2^2 + ... + p_2^{\alpha_2}\right)....\left(p_n^0 + p_n^1 + p_n^2 + ... + p_n^{\alpha_n}\right)$

(iv) The number of ways is which N can be resolved as a product of factors is

$$\begin{cases} \frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1)....(\alpha_k + 1), & \text{if N is not a perfect square} \\ \frac{1}{2}\left[(\alpha_1 + 1)(\alpha_2 + 1)....(\alpha_k + 1) + 1\right], & \text{if N is a perfect square} \end{cases}$$

(v) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or co-prime) to each other is equal to 2^{n-1} where n is the number of different factors in N.

The Pigeon-Hole Principle (PHP)

If more than n objects are distributed into n compartments. Some compartments must receive more than one object.

Formally, the pigeon hole principle states the following :

If $rn + 1$ pigeons $(r \geq 1)$ are distributed among n pigeon holes, one of the pigeon-holes will contain at least $r + 1$ pigeons.

Alternatively the pigeon hole principle states the following :

If *n* pigeons are placed in *m* pigeon-holes, then at least one pigeon hole will contain more than $\left[\frac{n-1}{m}\right]$ pigeons, where $[x]$ denotes the greatest integer $\leq x$

The Principle of inclusion and exclusion

If $A_1, A_2, ..., A_m$ are finite sets and $A = A_1 \cup A_2 \cup .... \cup A_m$, then

$$n(A) = a_1 - a_2 + a_3 - a_4 + ... + (-1)^{m+1} a_m \qquad \text{Where } a_1 = n(A_1) + n(A_2) + ... + n(A_m)$$

$$a_2 = \sum_{l \le i < j \le m} n(A_i \cap A_j),\ a_3 = \sum_{l \le i < j < k \le m} n(A_i \cap A_j \cap A_k) \text{ and so on.}$$

Corollary **(Sieve-Formula)** If $A_1, A_2, ..., A_m$ are m subset of a set A containing N elements, then

$$n(A'_1 \cap A'_2 \cap .... \cap A'_m)$$

$$= N - \sum_i n(A_i) + \sum_{1 \le i < j \le m} n(A_i \cap A_j) - \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k) + ... + (-1)^m n(A_1 \cap A_2 \cap .... \cap A_m)$$

Mathematical Induction Chapter 10

Let *P(n)* be a statement about natural numbers *n*

1. If $P(n)$ is true for $n=1$, and the truth of $P(n) \Rightarrow$ the truth of $P(n+1)$, then $P(n)$ is true for all natural numbers n;
2. If $P(n)$ is true for $n=1$ and the truth of $P(k)$ for all $n \le k \Rightarrow$ the truth of $P(k+1)$, then $P(n)$ is true for all natural numbers n ;
3. If $P(n)$ is true for $n=1$ and the falsity of $P(n+1) \Rightarrow$ the falsity of $P(n)$ is true for all natural numbers n.

REMARK :

1. Is called the first principle of induction;
2. Is called the second principle of induction;
3. Is called Fermation induction.

Let n be any integer and p a prime

1. $n^p - n$ is divisible by p;
2. If n is not divisible by p, then $n^{p-1}-1$ is divisible by p.

REMARK :

Statements (1) & (2) are equivalent. This result is called Fermat theorem.

For any prime P, $(P-1)!+1$ is divisible by P.

REMARK : The is wilson's theorem.

First Principle of Mathematical Induction

The statement $P(n)$ is true for all $n \in N$ if

1. $P(1)$ is true
2. $P(m)$ is true $\Rightarrow P(m+1)$ is true

The above statements can be generalized as $P(n)$ is true for all $n \in N$ and $n \ge k$ if

1. $P(k)$ is true
2. $P(m)$ is true $(m > k) \Rightarrow P(m+1)$ is true

Second Principle of Mathematical Induction

The statement $P(n)$ is true $\forall\ n \in N$, if

1. $P(1)$ and $P(2)$ are true
2. $P(m)$ and $P(m+1)$ are true $= P(m+2)$ is true

OR

The statement $P(n)$ is true $\forall\ n \in N$

if $(1) P(1) P(2) P(3)$ are true

$(2) P(m)$, $P(m+1)$, $P(m+2)$ are true $\Rightarrow P(m+3)$ is true

OR

The statement $P(n)$ is true $\forall\ n \in N$ if $P(1)$, $P(2)$, $P(3)$,....., $P(k)$ are true

$(2) P(m)$, $P(m+1)$, $P(m+2)$,.....$P(m+k)$ are true $\Rightarrow P(m+k+1)$ is true.

Binomial Theorem

Chapter **11**

BINOMIAL EXPRESSION

An algebraic expression consisting of two terms with + ve or – ve sign between them is called binomial expression.

For example : $(a+b)$, $(2x-3y)$, $\left(\frac{p}{x^2}-\frac{q}{x^4}\right)$, $\left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

The rule by which any power of binomial expression can be expanded is called the binomial theorem

If n is a positive integer and $x, y \in C$ then

$$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots\dots.. + {}^nC_r x^{n-r} y^r + \dots\dots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

$$i.e., (x+y)^n = \sum_{r=0}^{n} {}^nC_r . x^{n-r} . y^r \qquad \dots \text{(i)}$$

Here ${}^nC_0, {}^nC_1, {}^nC_2, \dots\dots {}^nC_n$ are called binomial coefficients and ${}^nC_r, = \frac{n!}{r!(n-r)!}$ for $0 \le r \le n$.

SOME IMPORTANT EXPANSIONS

1. Replacing y by $-y$ in (i), we get

$$(x-y)^n = {}^nC_0 x^{n-0} y^0 - {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 - \dots.. + (-1)^r\, {}^nC_r x^{n-r} y^r + \dots. + (-1)^n\, {}^nC_n x^o y^n$$

$$i.e.\ (x-y)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^{n-r} y^r$$

The terms in the expansion of $(x-y)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

2. Replacing x by 1 and y by x in equation (i) we get

$$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots.. + {}^nC_r x^r + \dots.. + {}^nC_n x^n \quad i.e., (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

This is expansion of $(1+x)^n$ in ascending power of x.

3. Replacing x by 1 and y by $-x$ in (i) we get ,

$$(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 - \dots.. + (-1)^r\, {}^nC_r x^r + \dots. + (-1)^n\, {}^nC_n x^n$$

$$i.e.\ (1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r$$

4. $(x+y)^n + (x-y)^n = 2\left[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \dots..\right]$ and

$$(x+y)^n - (x-y)^n = 2\left[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \dots..\right]$$

GENERAL TERM

The general term of the expansion is $(r+1)\,th$ term usually denoted by T_{r+1} and

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

Note :

(a) In the binomial expansion of $(x-y)^n, T_{r+1} = (-1)^r\, {}^nC_r x^{n-r} y^r$

(b) In the binomial expansion of $(1+x)^n$, $T_{r+1} = {}^nC_r x^r$

(c) In the binomial expansion of $(1-x)^n$, $T_{r+1} = (-1)^r\ {}^nC_r x^r$

(d) In the binomial expansion of $(x+y)^n$, the pth term from the end is $(n-p+2)\,th$ term from beginning.

MIDDLE TERM

The middle term depends upon the value of *n*.

1. **When *n* is even ,** then total number of terms in the expansion of $(x+y)^n$ is $n+1(\text{odd})$. So there is only one middle term $i.e. \left(\frac{n}{2}+1\right) th$ term is the middle term

$$T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}\, x^{n/2} y^{n/2}$$

2. **When *n* is odd,** then total number of terms in the expansion of $(x+y)^n$ is $n+1$ (even). So, there are two middle terms $i.e. \left(\frac{n+1}{2}\right) th$ and $\left(\frac{n+3}{2}\right) th$ are two middle terms.

$$T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}} y^{\frac{n-1}{2}} \quad \text{and} \quad T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$$

Note :

(a) When there are two middle terms in the expansion then their binomial coefficients are equal.

(b) Binomial coefficient of middle term is the greatest binomial coefficient.

PROPERTIES OF BINOMIAL COEFFICIENTS

In the binomial expansion of $(1+x)^n$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$$

Where ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ are the coefficients of various powers of *x* and called binomial coefficients, and they may be written as $C_0, C_1, C_2, ...C_n$.

Hence, $$(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_r x^r + ... + C_n x^n \qquad \text{… (i)}$$

1. The sum of binomial coefficients in the expansion of $(1+x)^n$ is 2^n

Putting $x=1$ in (i), we get, $2^n = C_0 + C_1 + C_2 + ... + C_n$ … (ii)

2. Sum of binomial coefficients with alternate signs is 0

Putting $x=-1$ in (i), we get, $C_0 - C_1 + C_2 - C_3 + ... = 0$ … (iii)

3. Sum of the coefficient of the odd terms in the expansion of $(1+x)^n$ is equal to sum of the coefficients of even terms and each is equal to 2^{n-1}

i.e. $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

4. ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}$ and so on.

5. Sum of product of coefficients in the expansion

$$C_0 C_r + C_1 C_{r+1} + ... + C_{n-r}.C_n = {}^{2n}C_{n+r} = \frac{2n!}{(n-r)!(n+r)!} \qquad \text{… (iv)}$$

6. Sum of squares of coefficients

Putting $r = 0$ in (iv), we get, ${}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

7. If ${}^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n.$

8. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

9. $n \cdot {}^{n-1}C_{r-1} = (n - r + 1)\, {}^nC_{r-1}$

10. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}$

11. $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + n.C_n = n.2^{n-1}$

12. $C_1 - 2C_2 + 3C_3 - \ldots\ldots\ldots = 0$

13. $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = (n+2)\,2^{n-1}$

14. $C_0^2 + C_1^2 + C_2^2 \ldots\ldots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$

15. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots. = \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2} . {}^nC_{n/2}. & \text{if n is even} \end{cases}$

MULTINOMIAL THEOREM FOR A POSITIVE INTEGRAL INDEX

If $x_1, x_2, \ldots, x_k$ are real numbers, then for all $n \in$ N,

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots\ldots x_k^{r_k},$$ where $r_1, r_2, \ldots r_k$ are all non-negative integers.

Note :

(a) The general term in the above expansion is $\dfrac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1}.x_2^{r_2}\ldots x_k^{r_k}$

(b) The total number of terms in the above expansion is

= number of non – negative integral solutions of the equation $r_1 + r_2 + \ldots + r_k = n$

= ${}^{n+k-1}C_n$ or ${}^{n+k-1}C_{k-1}$

(c) Coefficient of $x_1^{r_1}.x_2^{r_2}\ldots x_k^{r_k}$ in the expansion of $(a_1x_1 + a_2x_2 + \ldots + a_kx_k)^n$

$= \dfrac{n!}{r_1!r_2!\ldots r_k!} a_1^{r_1}.a_2^{r_2}\ldots a_k^{r_k}$

(d) Greatest coefficient in the expansion of $(x_1 + x_2 + \ldots + x_k)^n$

$= \dfrac{n!}{(q!)^{k-r}.\left[(q+1)!\right]^r}$ where q is the quotient and r the remainder when n is divided by k.

(e) Sum of all the coefficient is obtained by putting all the variables x_i equal to 1 and is n^m.

BINOMIAL THEOREM FOR ANY INDEX

If n and x are real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \text{ to } \infty \quad \ldots(1)$$

Note :

1. In the above expansion, the first term must be unity. In the expansion of $(a+x)^n$, where n is either a negative integer or a fraction or an irrational number, we proceed as follows :

$$(a+x)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left[1+n.\frac{x}{a}+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+...\right]$$

and the expansion is valid when $\left|\frac{x}{a}\right|<1$ *i.e.* $|x|<|a|$.

2. There are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.
3. If x is so small that its square and higher powers may be neglected, then approximate value of

$$(1+x)^n = 1+nx.$$

GENERAL TERM IN THE EXPANSION $(1+x)^n$

The $(r+1)$th term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r$$

SOME IMPORTANT DEDUCTIONS

1. Replacing n by $-n$ in (1), we get

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+...+(-1)^r\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r+...$$

General Term : $T_{r+1} = (-1)^r\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r$

2. Replacing x by $-x$ in (1), we get

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+...+(-1)^r\frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r+...$$

General Term : $T_{r+1} = (-1)^r\frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r$

3. Replacing x by $-x$ and n by $-n$ in (1), we get

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+...+\frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r+...$$

General Term : $T_{r+1} = \frac{n(n+1)(n+2)...(n+r-1)}{r!}x^r$

NUMERICALLY GREATEST TERM IN THE EXPANSION OF (1+*x*)^n :

Case 1: '*n*' is + ve integer

If $\left(\frac{n+1}{1+|x|}\right)|x|$ = an integer denote it by p; then $T_{p+1} = T_p$ both are greatest.

If $\left(\frac{n+1}{1+|x|}\right)|x| = \underbrace{\text{integer}}_{p}$ + proper fraction, then T_{p+1} is greatest term

Case 2: '*n*' be a + ve fraction or – ve integer

(a) If $|x|$ = greater than unity, there is no greatest term

(b) If $|x|$ = less than unity; there will be a greatest term

If $\frac{(n+1)}{1+|x|}\,|x| =$ an $\underbrace{\text{integer}}_{P}$, then $T_{p+1}=T_p$ both are greatest term

If $\left(\frac{n+1}{1+|x|}\right)|x| = \underbrace{\text{integer}}_{p}$ + proper fraction, then T_{p+1} is greatest term

AN IMPORTANT THEOREM

If $\left(\sqrt{A}+B\right)^n = I+f$ where I and n are positive integers, If n being odd and $0 \le f < 1$ then

$(1+f)\cdot f = K^n$ where $A-B^2=K>0$ and $\sqrt{A}-B<1$.

If n is even integer then $\left(\sqrt{A}+B\right)^n+\left(\sqrt{A}-B\right)^n = 1+f+f'$

Hence L. H.S. and I are integers

$\therefore$ $f+f'$ is also integer $\Rightarrow$ $f+f'=1$; $\therefore$ $f'=(1-f)$,

Hence $(I+f)(1-f) = (I+f)f' = \left(\sqrt{A}+B\right)^n\left(\sqrt{A}-B\right)^n = \left(A-B^2\right)^n = K^n$.

SOME IMPORTANT EXPANSIONS

(i) $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+.....+\frac{n(n-1)(n-2).....(n-r+1)}{r!}x^r+.....$

(ii) $(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-.....+\frac{n(n-1)(n-2)....(n-r+1)}{r!}(-x)^r+.....$

(iii) $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+....+\frac{n(n+1)....(n+r-1)}{r!}x^r+...$

(iv) $(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+....+\frac{n(n+1).....(n+r-1)}{r!}(-x)^r+.....$

(v) $(1+x)^{-p/q} = 1-\frac{p}{q}x+\frac{p(p+q)}{q\cdot 2q}x^2-\frac{p(p+q)(p+2q)}{q\cdot 2q\cdot 3q}x^3+.......$

(vi) $(1-x)^{-p/q} = 1+\frac{p}{q}x+\frac{p(p+q)}{q\cdot 2q}x^2+\frac{p(p+q)(p+2q)}{q\cdot 2q\cdot 3q}x^3+.....$

(vii) $(1+x)^{-1} = 1-x+x^2-x^3+.......+(-1)^r x^r+.....$

(viii) $(1-x)^{-1} = 1+x+x^2+x^3+..........+x^r+.......$

(ix) $(1+x)^{-2} = 1-2x+3x^2-4x^3+.......+(-1)^r(r+1)x^r+......$

(x) $(1-x)^{-2} = 1+2x+3x^2+4x^3+......+(r+1)x^r+.....$

(xi) $(1+x)^{-3} = 1-3x+6x^2-10x^3+.............+\frac{(r+1)(r+2)}{2}\cdot(-1)^r x^r+......$

(xii) $(1-x)^{-3} = 1+3x+6x^2+10x^3+...............+\frac{(r+1)(r+2)}{2}x^r+......$

(xiii) $(1+x)^{-n} = {}^{1-n}C_1x+{}^{n+1}C_2x^2.....$ where $n\in N$

(xiv) $(1-x)^{-n} = {}^{1+n}C_1x+{}^{n+1}C_2x^2+{}^{n+2}C_3x^3.....$ where $n\in N$

Trigonometric Ratios, Identities & Equations

Chapter 12

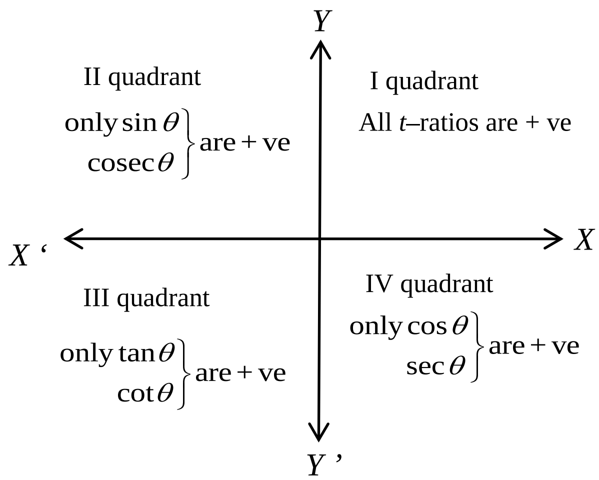
BASIC FORMULAE

1. $\sin^2 A+\cos^2 A=1$ or $\cos^2 A=1-\sin^2 A$ or $\sin^2 A=1-\cos^2 A.$

2. $1+\tan^2 A=\sec^2 A$ or $\sec^2 A-\tan^2 A=1.$ 3. $1+\cot^2 A=\mathrm{cosec}^2 A,$ or $\mathrm{cosec}^2 A-\cot^2 A=1$

4. $\sin A\cos ecA=\tan A\cot A=\cos A\sec A=1.$

A system of rectangular coordinate axes divides a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The values of the trigonometric ratios in the various quadrants are shown in Fig.



Formulae for the trigonometric ratios of sum and differences of two angles

1. $\sin\left(A+B\right)=\sin A\cos B+\cos A\sin B$

2. $\sin\left(A-B\right)=\sin A\cos B-\cos A\sin B$

3. $\cos\left(A+B\right)=\cos A\cos B-\sin A\sin B$

4. $\cos\left(A-B\right)=\cos A\cos B+\sin A\sin B$

5. $\tan\left(A+B\right)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ 6. $\tan\left(A-B\right)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

7. $\cot\left(A+B\right)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$ 8. $\cot\left(A-B\right)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

9. $\sin\left(A+B\right)\sin\left(A-B\right)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

10. $\cos\left(A+B\right)\cos\left(A-B\right)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

11. $\tan A\pm\tan B=\dfrac{\sin A}{\cos A}\pm\dfrac{\sin B}{\cos B}=\dfrac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B} = \dfrac{\sin\left(A\pm B\right)}{\cos A.\cos B},$

12. $\cot A\pm\cot B=\dfrac{\sin\left(B\pm A\right)}{\sin A\sin B}$

13. $\tan A+\cot B=\dfrac{\cos\left(B-A\right)}{\cos A\sin B}$

14. $\tan A - \cot B = \dfrac{-\cos(B+A)}{\cos A \sin B}$

Formulae for the trigonometric ratios of sum and differences of three angle

1. $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

   or $\sin(A+B+C) = \cos A \cos B \cos C$ $(\tan A + \tan B + \tan C - \tan A. \tan B. \tan C)$

2. $\cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$

   $\cos(A+B+C) = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$

3. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

4. $\cot(A+B+C) = \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot A \cot B + \cot B \cot C + \cot C . \cot A - 1} = \dfrac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$

In general

5. $\sin(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 ... \cos A_n (S_1 - S_3 + S_5 - S_7 + ...)$

6. $\cos(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 .... \cos A_n (1 - S_2 + S_4 - S_6 ....)$

7. $\tan(A_1 + A_2 + ... A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$

   where, $S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.

   $S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan_3 + ... =$ The sum of the tangents taken two at time.

   $S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \cdot \tan A_3 \tan A_4 + ...$ = Sum of tangents three at a time, and so on.

   If $A_1 = A_2 = ... = A_n = A$, *then* $S_1 = n \tan A$

   $S_2 = {}^nC_2 \tan^2 A, S_3 = {}^nC_3 \tan^3 A, ...$

8. $\sin nA = \cos^n A\left({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ...\right)$

9. $\cos nA = \cos^n A\left(1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - ...\right)$

10. $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - ...}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A - {}^nC_6 \tan^6 A + ...}$

11. $\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... + \sin(\alpha+(n-1)\beta) = \dfrac{\sin\left\{\alpha + (n-1)\left(\beta/2\right)\right\}.\sin\left(n\beta/2\right)}{\sin\left(\beta/2\right)}$

12. $\cos(\alpha) + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ... + \cos(\alpha+(n-1)\beta) = \dfrac{\cos\left\{\alpha + (n-1)\left(\beta/2\right)\right\}.\sin\left(n\beta/2\right)}{\sin\left(\beta/2\right)}$

Formulae to transform the product into sum or difference

1. $2\sin A \cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A \sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A \cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Let $A+B=C$ *and* $A-B=D$ Then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$

Formulae to transform the sum or difference into product

1. $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
2. $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
3. $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
4. $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Trigonometric ratio of multiple of an angle

1. $\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A=2\cos^2 A-1=1-2\sin^2 A = \cos^2 A-\sin^2 A=\frac{1-\tan^2 A}{1+\tan^2 A}$.
3. $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A=3\sin A-4\sin^3 A=4\sin\left(60^o-A\right).\sin A.\sin\left(60^o+A\right)$
5. $\cos 3A=4\cos^3 A-3\cos A=4\cos\left(60^o-A\right).\cos A.\cos\left(60^o+A\right)$
6. $\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\tan\left(60^o-A\right).\tan A.\tan\left(60^o+A\right)$
7. $\sin 4\theta=4\sin\theta.\cos^3\theta-4\cos\theta\sin^3\theta$
8. $\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$
9. $\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$
10. $\sin 5A=16\sin^5 A-20\sin^3 A+5\sin A$
11. $\cos 5A=16\cos^5 A-20\cos^3 A+5\cos A$

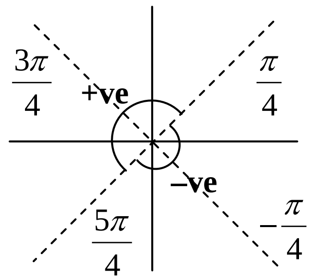
Trigonometrical values

1. $\sin 22\frac{1}{2}^\circ=\cos 67\frac{1}{2}^\circ=\frac{1}{2}\sqrt{2-\sqrt{2}}$
2. $\sin 67\frac{1}{2}^\circ=\cos 22\frac{1}{2}^\circ=\frac{1}{2}\sqrt{2+\sqrt{2}}$
3. $\sin 15^\circ=\cos 75^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$
4. $\cos 15^\circ=\sin 75^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}$
5. $\tan 15^\circ=\cot 75^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$
6. $\tan 75^\circ=\cot 15^\circ=\frac{\sqrt{3}+1}{\sqrt{3}-1}=2+\sqrt{3}$
7. $\tan\left(22\frac{1}{2}\right)^\circ=\cot\left(67\frac{1}{2}\right)^\circ=\sqrt{2}-1$
8. $\tan\left(67\frac{1}{2}\right)^\circ=\cot\left(22\frac{1}{2}\right)^\circ=\sqrt{2}+1$
9. $\sin 18^\circ=\frac{\sqrt{5}-1}{4}=\cos 72^\circ$
10. $\sin 36^\circ=\frac{\sqrt{10-2\sqrt{5}}}{4}=\cos 54^\circ$
11. $\sin 54^\circ=\frac{\sqrt{5}+1}{4}=\cos 36^\circ$
12. $\sin 72^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cos 18^\circ$

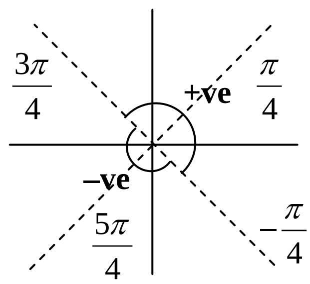
CONDITIONAL TRIGONOMETRICAL IDENTITIES

If $A+B+C=180^o$, then

1. $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$
2. $\sin 2A+\sin 2B-\sin 2C = 4\cos A\cos B\sin C$
3. $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$
4. $\cos 2A+\cos 2B-\cos 2C = 1-4\sin A\sin B\cos C$
5. $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
6. $\sin A+\sin B-\sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
7. $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{C}{2}$
8. $\cos A+\cos B-\cos C = -1+4\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}$
9. (i) $\sin^2 A+\sin^2 B-\sin^2 C = 2\sin A\sin B\cos C$
   (ii) $\cos^2 A+\cos^2 B-\cos^2 C = 1-2\sin A\sin B\cos C$
   (iii) $\cos^2 A+\cos^2 B+\cos^2 C = 1-2\cos A\cos B\cos C$
   (iv) $\sin^2 A+\sin^2 B+\sin^2 C = 2+2\cos A\cos B\cos C$
10. (i) $\tan A+\tan B+\tan C = \tan A\tan B\tan C$
   (ii) $\cot B\cot C+\cot C\cot A+\cot A.\cos B = 1$
   (iii) $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}.\tan\frac{B}{2} = 1$
   (iv) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$



$\sin\theta-\cos\theta$ is +ve or –ve in interval shown above



$\sin\theta+\cos\theta$ is +ve or –ve in interval shown above

TIPS AND TRICKS

1. $\sin n\pi = 0$
2. $\cos n\pi = (-1)^n$
3. $\sin(n\pi\pm\theta) = \pm(-1)^n\sin\theta$
4. $\cos(n\pi\pm\theta)=(-1)^n\cos\theta$
5. $\tan(n\pi\pm\theta)=\pm\tan\theta$
6. $\sin\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n-1}{2}}\cos\theta, & \text{if } n \text{ is odd}\\ (-1)^{n/2}\sin\theta, & \text{if } n \text{ is even}\end{cases}$
7. $\cos\left(\frac{n\pi}{2}+\theta\right) = \begin{cases}(-1)^{\frac{n+1}{2}}\sin\theta, & \text{if } n \text{ is odd}\\ (-1)^{n/2}\cos\theta, & \text{if } n \text{ is even}\end{cases}$
8. $\tan\frac{A}{2} = \frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$

9. $\prod_{r=0}^{n-1} \cos 2^r A = \dfrac{\sin 2^n A}{2^n \sin A}$

10. $\tan\alpha = \cot\alpha - 2\cot 2\alpha, \quad \operatorname{cosec} 2\alpha = \cot\alpha - \cot 2\alpha$

11. $\sec\alpha.\sec(\alpha+\beta) = \operatorname{cosec}\beta\{\tan(\alpha+\beta) - \tan\alpha\}$

12. $\tan\alpha.\tan(\alpha+\beta) = \cot\beta\{\tan(\alpha+\beta) - \tan\alpha\} - 1$

13. $\operatorname{cosec}\alpha\operatorname{cosec}(\alpha+\beta) = \operatorname{cosec}\beta\{\cot\alpha - \cot(\alpha+\beta)\}$

14. $\tan\alpha.\sec 2\alpha = \tan 2\alpha - \tan\alpha$

15. $\tan^2\alpha.\tan 2\alpha = \tan 2\alpha - 2\tan\alpha$

16. $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A\cos B}$

17. $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A\cos B}$

18. $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A\sin B}$

19. $\cot A - \cot B = \dfrac{\sin(B-A)}{\sin A\sin B}$

20. $1 + \tan A\tan B = \dfrac{\cos(A-B)}{\cos A\cos B}$

21. $1 - \tan A\tan B = \dfrac{\cos(A+B)}{\cos A\cos B}$

22. $1 + \cot A\cot B = \dfrac{\cos(A-B)}{\sin A\sin B}$

23. $1 - \cot A\cot B = \dfrac{-\cos(A+B)}{\sin A\sin B}$

24. If $\alpha+\beta+\gamma = 0$ then

(a) $\sin\alpha + \sin\beta + \sin\gamma = -4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$, (b) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

(c) $\cos\alpha + \cos\beta + \cos\gamma = -1 + 4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$

24. $\tan\theta + \cot\theta = 2\operatorname{cosec}2\theta$ 25. $\cot\theta - \tan\theta = 2\cot 2\theta$

26. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

27. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

28. $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$

29. $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\beta+\gamma}{2}\sin\dfrac{\gamma+\alpha}{2}$

30. $\tan\alpha + \tan\beta + \tan\gamma - \tan(\alpha+\beta+\gamma) = \tan\alpha\tan\beta\tan\gamma\left[1 - \dfrac{\cot\alpha + \cot\beta + \cot\gamma}{\cot(\alpha+\beta+\gamma)}\right]$

31. For any α, β and γ we have the following identities.

(a) $\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta) = 0$

(b) $\cos\alpha\sin(\beta-\gamma)+\cos\beta\sin(\gamma-\alpha)+\cos\gamma\sin(\alpha-\beta) = 0$.

32. (a) $(\cos x+\sin x)(\cos y+\sin y)=\cos(x-y)+\sin(x+y)$

(b) $(\cos x-\sin x)(\cos y-\sin y)=\cos(x-y)-\sin(x+y)$

TRIGONOMETRICAL EQUATIONS WITH THEIR GENERAL SOLUTION.

Trigonometrical equation	**General solution**
$\sin\theta=\sin\alpha$	$\theta=n\pi+(-1)^n\alpha$ where $n\in I$
$\cos\theta=\cos\alpha$	$\theta=2n\pi\pm\alpha$ where $n\in I$
$\tan\theta=\tan\alpha$	$\theta=n\pi+\alpha$ where $n\in I$
$\sin^2\theta=\sin^2\alpha$ $\tan^2\theta=\tan^2\alpha$ $\cos^2\theta=\cos^2\alpha$	$\theta=n\pi\pm\alpha$ where $n\in I$

But it is better to remember the following results instead of using above formulae in the following cases.

Trigonometrical equation	**General solution**
$\sin\theta=0$	$\theta=n\pi$ where $n\in I$
$\cos\theta=0$	$\theta=n\pi+\pi/2$ where $n\in I$
$\sin\theta=1$	$\theta=2n\pi+\pi/2$ where $n\in I$
$\sin\theta=-1$	$\theta=2n\pi-\pi/2$ where $n\in I$
$\cos\theta=1$	$\theta=2n\pi$ where $n\in I$
$\cos\theta=-1$	$\theta=(2n+1)\pi$ where $n\in I$
$\sin\theta=\pm1$	$\theta=(2n+1)\pi/2$ where $n\in I$
$\cos\theta=\pm1$	$\theta=n\pi$ where $n\in I$

GENERAL SOLUTION OF THE FORM $a\cos\theta+b\sin\theta=c$

In $a\cos\theta+b\sin\theta=c$, put $a=r\cos\alpha$ and $b=r\sin\alpha$ where $r=\sqrt{a^2+b^2}$ and $|c|\le\sqrt{a^2+b^2}$.

Then, $r(\cos\alpha\cos\theta+\sin\alpha\sin\theta)=c \Rightarrow \cos(\theta-\alpha)=\dfrac{c}{\sqrt{a^2+b^2}}=\cos\beta$, (say)

$\Rightarrow \theta-\alpha=2n\pi\pm\beta \Rightarrow \theta=2n\pi\pm\beta+\alpha$, where $\tan\alpha=\dfrac{b}{a}$ is the general solution.

Inverse Trigonometric Function

Chapter 13

INVERSE FUNCTION

Let f be a function defined from a set A to a set B, i.e. $f : A \to B$ and g be a function defined from the set B to the set A, i.e., $g : B \to A$; then the function g is said to be inverse of f if

$$g\left\{f\left(x\right)\right\} = x,\ \forall x \in A \text{ and the function } g \text{ is denoted by } f^{-1}.$$

Properties of inverse of a function :

(i) The inverse of bijection is unique.

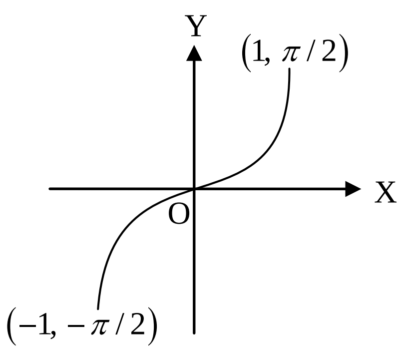
(ii) If $f : A \to B$ is bijection and $g : B \to A$ is inverse of f, then

$$f\ o\ g = I_B \text{ and } g\ o\ f = I_A$$

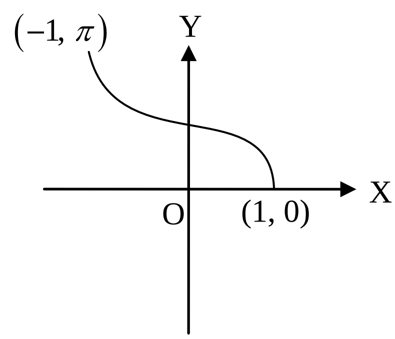
where, I_A and I_B are identity functions on the sets A and B respectively.

Graphs of inverse trigonometric functions

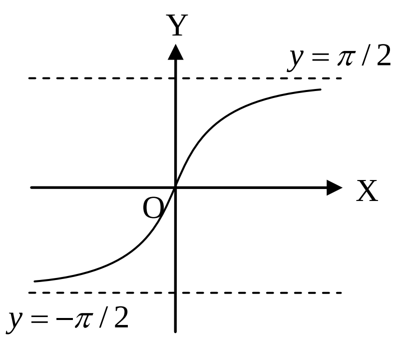
(i) Graph of $y = \sin^{-1} x$



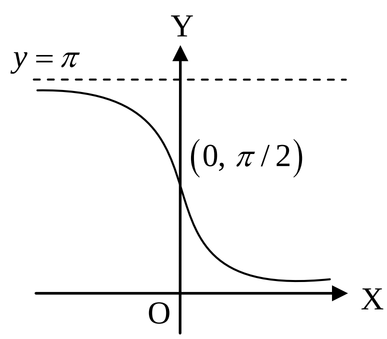
(ii) Graph of $y = \cos^{-1} x$



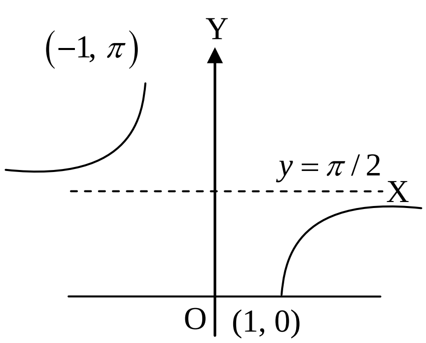
(iii) Graph of $y = \tan^{-1} x$



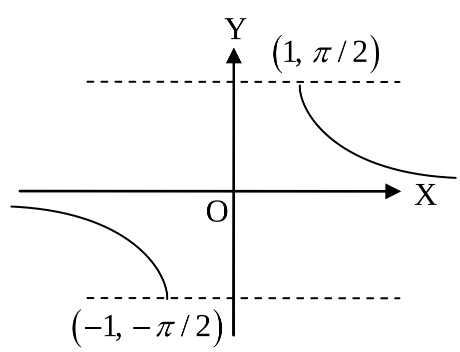
(iv) Graph of $y = \cot^{-1} x$



(v) Graph of $y = \sec^{-1} x$



(vi) Graph of $y = \text{cosec}^{-1} x$



DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

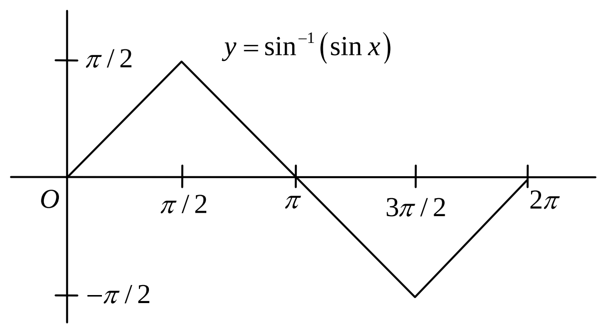
Function	**Domain (*D*)**	**Range (*R*)**
$\sin^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$0 \le \theta \le \pi$ or $[0, \pi]$
$\tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$x \in R$	$0 < \theta < \pi$ or $(0, \pi)$
$\sec^{-1} x$	$x \le -1$, or $1 \le x$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne \frac{\pi}{2}$, $0 \le \theta \le \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\operatorname{cosec}^{-1} x$	$x \le -1$, or $1 \le x$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne 0$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. (i) $\sin^{-1}(\sin\theta) = \theta$ if and only if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

$$\sin^{-1}(\sin x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

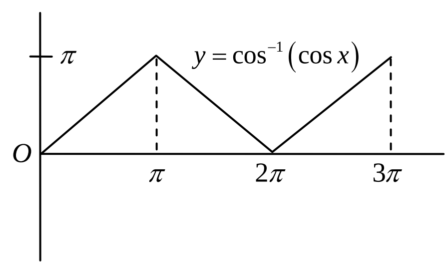
$\Rightarrow$ $f(x) = \sin^{-1}(\sin x)$ is periodic with period 2π.



(ii) $\cos^{-1}(\cos\theta) = \theta$ if and only if $0 \le \theta \le \pi$ and

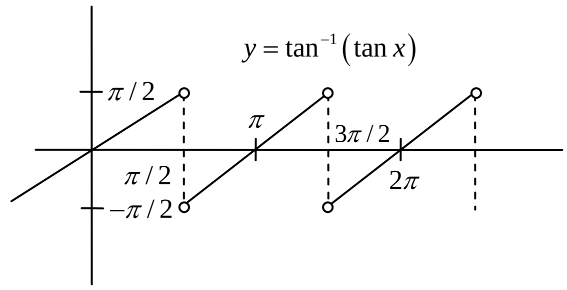
$$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$$

$\Rightarrow$ $f(x) = \cos^{-1}(\cos x)$ is periodic with period 2π.



(iii) $\tan^{-1}(\tan\theta)=\theta$ if and only if $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and

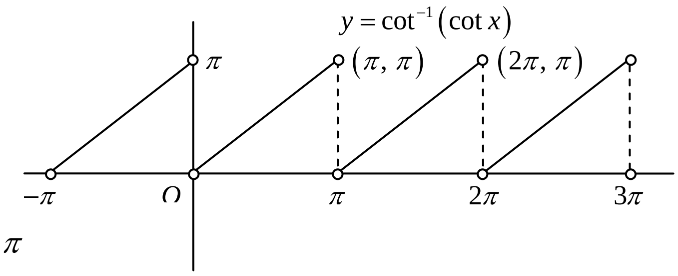
$$\tan^{-1}(\tan x)=\begin{cases} x, & 0\le x<\frac{\pi}{2} \\ x-\pi, & \frac{\pi}{2}<x<\frac{3\pi}{2} \\ x-2\pi, & \frac{3\pi}{2}<x\le 2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\tan^{-1}(\tan x)$ is periodic with period π.

(iv) $\cot^{-1}(\cot\theta)=\theta$ if and only if $0<\theta<\pi$ and

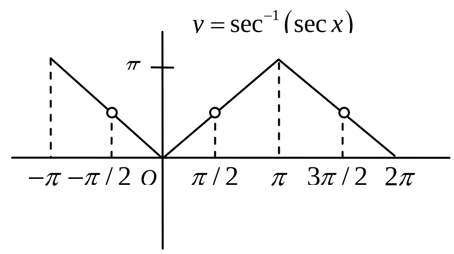
$$\cot^{-1}(\cot x)=\begin{cases} x+\pi, & -\pi<x<0 \\ x, & 0<x<\pi \\ x-\pi, & \pi<x<2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\cot^{-1}(\cot x)$ is periodic with period π

(v) $\sec^{-1}(\sec\theta)=\theta$ if and only if $0\le\theta<\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$ and

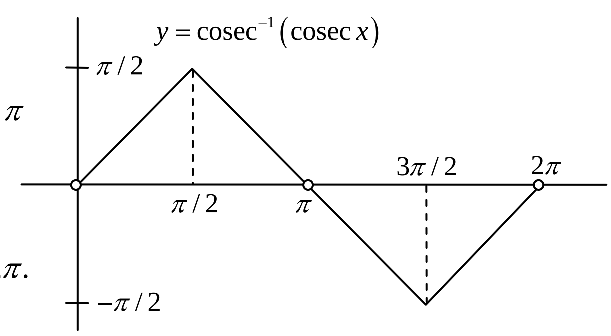
$$\sec^{-1}(\sec x)=\begin{cases} -x, & -\pi\le x\le 0, \quad x\ne-\frac{\pi}{2} \\ x, & 0<x\le\pi, \quad x\ne\frac{\pi}{2} \end{cases}$$



$\Rightarrow$ $f(x)=\sec^{-1}(\sec x)$ is periodic with period 2π.

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta$ if and only if $-\frac{\pi}{2}\le\theta<0$ or $0<\theta\le\frac{\pi}{2}$ and

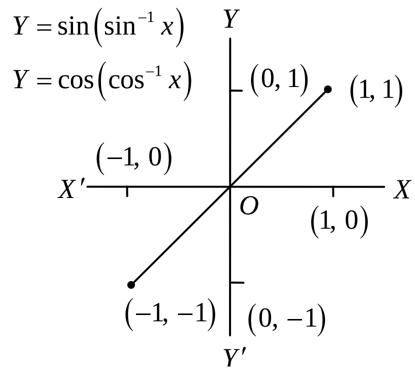
$$\operatorname{cosec}^{-1}(\operatorname{cosec} x)=\begin{cases} x, & 0<x\le\frac{\pi}{2}, \\ \pi-x, & \frac{\pi}{2}<x\le\frac{3\pi}{2} \quad x\ne\pi \\ x-2\pi, & \frac{3\pi}{2}<x<2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is periodic with period 2π.

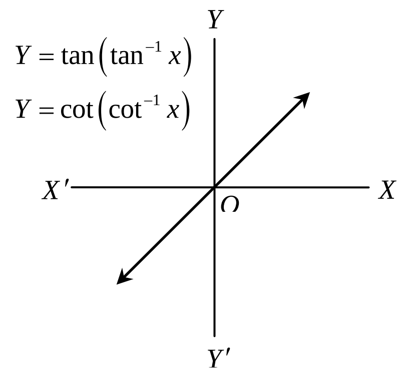
(vii) (a) $\sin\left(\sin^{-1} x\right) = x$ iff $-1 \le x \le 1$

(b) $\cos\left(\cos^{-1} x\right) = x$ iff $-1 \le x \le 1$



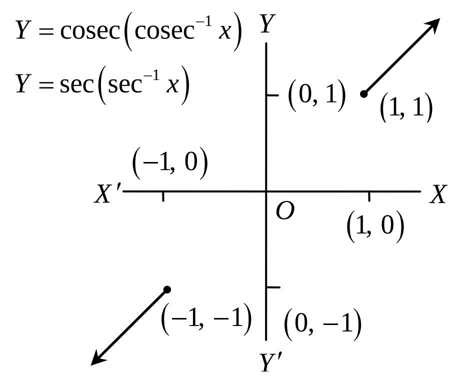
(viiii) (a) $\tan\left(\tan^{-1} x\right) = x$ for all x

(b) $\cot\left(\cot^{-1} x\right) = x$ for all x



(ix) (a) $\sec\left(\sec^{-1} x\right) = x$ iff $x \ge 1$ or $x \le -1$

(b) $\operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right) = x$ iff $x \ge 1$ or $x \le -1$



2. (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$,

3. $$\begin{cases} \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in [-1, 1] \\ \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in R \\ \sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

4. **Principal values for inverse circular functions.**

Principal values for $x \geq 0$	**Principal values for** $x < 0$
$0 \leq \sin^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1} x < 0$
$0 \leq \cos^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \leq \pi$
$0 \leq \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \leq \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \leq \pi$
$0 < \operatorname{cosec}^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \operatorname{cosec}^{-1} x < 0$

5. **Conversion property :**

(i) $$\begin{cases} \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}, & 0 \leq x \leq 1 \\ \sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}, & -1 \leq x \leq 0 \end{cases}$$

(ii) $$\begin{cases} \sin^{-1} x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}, & 0 < x \leq 1 \\ \sin^{-1} x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} - \pi, & -1 \leq x < 0 \end{cases}$$

(iii) $$\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad |x| < 1$$

(iv) $$\begin{cases} \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}, & 0 \leq x \leq 1 \\ \cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2}, & -1 \leq x \leq 0 \end{cases}$$

(v) $$\begin{cases} \cos^{-1} x = \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}, & 0 < x \leq 1 \\ \cos^{-1} x = \pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x}, & -1 \leq x < 0 \end{cases}$$

(vi) $$\cos^{-1} x = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad |x| < 1$$

(vii) $$\begin{cases} \tan^{-1} x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}, & x \geq 0 \\ \tan^{-1} x = -\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}, & x \leq 0 \end{cases}$$

(viii) $\begin{cases} \tan^{-1} x = \cot^{-1} \frac{1}{x}, & x > 0 \\ \tan^{-1} x = \cot^{-1} \frac{1}{x} - \pi, & x < 0 \end{cases}$

(ix) $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}} \quad \forall\ x \in R$

(x) $\begin{cases} \cot^{-1} x = \sin^{-1} \frac{1}{\sqrt{1+x^2}}, & x \geq 0 \\ \cot^{-1} x = \pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}}, & x < 0 \end{cases}$

(xi) $\begin{cases} \cot^{-1} x = \tan^{-1} \frac{1}{x}, & x > 0 \\ \cot^{-1} x = \pi + \tan^{-1} \frac{1}{x}, & x < 0 \end{cases}$

(xii) $\cot^{-1} x = \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \forall x \in R$

6. **General values of inverse circular functions :** We know that if α is the smallest angle whose sine is x, then all the angles whose sine is x can be written as $n\pi + (-1)^n \alpha$, where $n \in I$. Therefore, the general value of $\text{Sin}^{-1}x$ can be taken as $n\pi + (-1)^n \alpha$.

Thus, we have $\text{Sin}^{-1}x = n\pi + (-1)^n \alpha,\ -1 \leq x \leq 1$ if $\sin\alpha = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

Similarly, general values of other inverse circular functions are given as follows :

$\text{Cos}^{-1}x = 2n\pi \pm \alpha,\ -1 \leq x \leq 1$; If $\cos\alpha = x,\ 0 \leq \alpha \leq \pi$

$\text{Tan}^{-1}x = n\pi + \alpha,\ x \in R$; If $\tan\alpha = x,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\text{Cot}^{-1}x = n\pi + \alpha,\ x \in R$ If $\cot\alpha = x,\ 0 < \alpha < \pi$

$\text{Sec}^{-1}x = 2n\pi \pm \alpha,\ x \leq -1$ or $x \geq 1$ If $\sec\alpha = x,\ 0 \leq \alpha \leq \pi$ and $\alpha \neq \frac{\pi}{2}$

$\text{Cosec}^{-1}x = n\pi + (-1)^n \alpha,\ x \leq -1$ or $x \geq 1$ If $\text{cosec}\ \alpha = x,\ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ and $\alpha \neq 0$

Note : The first letter in all above inverse Trigonometric function are CAPITAL LETTER

Formulae for sum, difference of inverse trigonometric function

(1) $\begin{cases} \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \geq 1 \end{cases}$

(2) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}; \quad x \geq 0,\ y \geq 0$

(3) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}; \qquad x \geq 0,\ y \geq 0$

(4) $$\begin{cases} \cos^{-1} x - \cos^{-1} y = \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x \geq 0,\ y \geq 0,\ x \leq y \\ \cos^{-1} x - \cos^{-1} y = -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x \geq 0,\ y \geq 0,\ x \geq y \end{cases}$$

(5) $$\begin{cases} \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x \geq 0,\ y \geq 0 \text{ and } xy < 1 \\ \tan^{-1} x + \tan^{-1} y = \pi/2\ ; & x > 0,\ y > 0 \text{ and } xy = 1 \\ \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x \geq 0,\ y \geq 0 \text{ and } xy > 1 \end{cases}$$

(6) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); \qquad x \geq 0,\ y \geq 0$

Inverse trigonometric ratios of multiple angles

1. $$\begin{cases} 2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{1}{\sqrt{2}} \\ 2\sin^{-1} x = \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } \dfrac{1}{\sqrt{2}} < x \leq 1 \\ 2\sin^{-1} x = -\pi + \sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -1 \leq x \leq \dfrac{-1}{\sqrt{2}} \end{cases}$$

2. $$\begin{cases} 3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), & \text{If } \dfrac{-1}{2} \leq x \leq \dfrac{1}{2} \\ 3\sin^{-1} x = \pi - \sin^{-1}\left(3x - 4x^3\right), & \text{If } \dfrac{1}{2} < x \leq 1 \\ 3\sin^{-1} x = -\pi - \sin^{-1}\left(3x - 4x^3\right), & \text{If } -1 \leq x < -\dfrac{1}{2} \end{cases}$$

3. $$\begin{cases} 2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right), & \text{If } 0 \leq x \leq 1 \\ 2\cos^{-1} x = 2\pi - \cos^{-1}\left(2x^2 - 1\right), & \text{If } -1 \leq x \leq 0 \end{cases}$$

4. $$\begin{cases} 3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), & \text{If } \dfrac{1}{2} \leq x \leq 1 \\ 3\cos^{-1} x = 2\pi - \cos^{-1}\left(4x^2 - 3x\right), & \text{If } -\dfrac{1}{2} \leq x \leq \dfrac{1}{2}\ , \\ 3\cos^{-1} x = 2\pi + \cos^{-1}\left(4x^3 - 3x\right), & \text{If } -1 \leq x \leq -\dfrac{1}{2} \end{cases}$$

5. $$\begin{cases} 2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } -1 < x < 1 \\ 2\tan^{-1} x = \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1} x = -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } x < -1 \end{cases}$$

6. $$\begin{cases} 2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } -1 \le x \le 1 \\ 2\tan^{-1} x = \pi + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1} x = -\pi + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } x < -1 \end{cases}$$

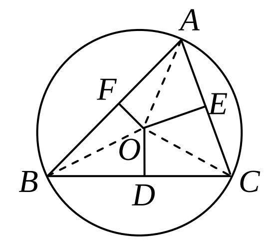
7. $$\begin{cases} 2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{If } 0 \le x \\ 2\tan^{-1} x = -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{If } x \le 0 \end{cases}$$

8. $$\begin{cases} 3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x = \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } x > \dfrac{1}{\sqrt{3}} \\ 3\tan^{-1} x = -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

Properties & Solution of Triangle, Height & Distance

Chapter 14

THE LAW OF SINES OR SINE RULE



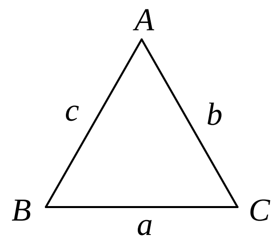
The sides of a triangle are proportional to the sines of the angles opposite to them

i.e., $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, (say)

More generally, if R be the radius of the circumcircle of the triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

THE LAW OF COSINES OR COSINE RULE

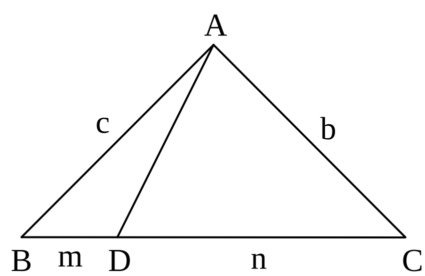


1. $a^2 = b^2 + c^2 - 2bc\cos A \;\; \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$
2. $b^2 = c^2 + a^2 - 2ca\cos B \;\; \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ca}$
3. $c^2 = a^2 + b^2 - 2ab\cos C \;\; \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

PROJECTION FORMULAE

1. $a = b\cos C + c\cos B$
2. $b = c\cos A + a\cos C$
3. $c = a\cos B + b\cos A$

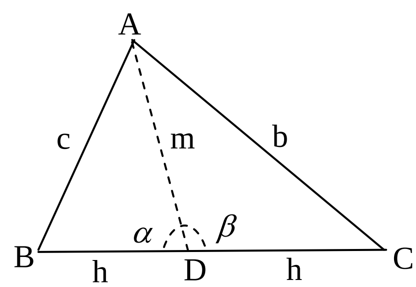
APOLLONIUS THEOREM



(a) $(m+n)AD^2 = mb^2 + nc^2 - mCD^2 - nBD^2$

(b) $(m+n)^2 AD^2 = (m+n)(mb^2 + nc^2) - a^2mn$

Apollonius theorem for medians



In every triangle the sum of the squares of any two sides is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.

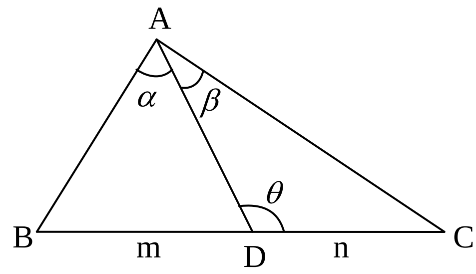
For any triangle ABC, $b^2 + c^2 = 2(h^2 + m^2) = 2\left\{m^2 + \left(a/2\right)^2\right\}$ by use of cosine rule.

If $\therefore\ \Delta$ be right angled , the mid point of hypotenuse is equidistant from the three vertices so that $DA = DB = DC$.

$\therefore\ b^2 + c^2 = a^2$ which is Pythagoras theorem. This theorem is very useful for solving problems of height and distance.

ANGLE BETWEEN MEDIAN AND THE SIDE OF A TRIANGLE

$$\sin\theta = \frac{2b}{\sqrt{2b^2+2c^2-a^2}}\sin C$$



THE $m-n$ Rule

If the triangle ABC, point D divides BC in the ratio $m:n$, and $\angle ADC = \theta$, then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$;

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$.

NAPIER'S ANALOGY (LAW OF TANGENTS)

For any triangle ABC,

(i) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ (ii) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

MOLLWEIDE'S FORMULA

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \qquad \frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

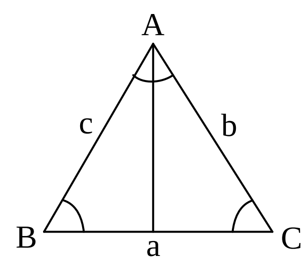
AREA OF TRIANGLE

Let three angles of $a\Delta$ be denoted by A, B, C and the sides opposite to these angles by letters a, b, and c respectively

1. **When two sides and the included angle be given:**

The area of triangle ABC is given by

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$



i.e. $\Delta = \frac{1}{2}(\text{Product of two side})\times\text{sine of included angle}$

2. **When three sides are given**

Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

where semi perimeter of triangle $s = \frac{a+b+c}{2}$.

3. **When three sides and the circum – radius be given** :

Area of triangle $\Delta = \frac{abc}{4R}$, where R be the circum-radius of the triangle.

4. **When two angles and included side be given** :

$$\Delta = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)} = \frac{1}{2}b^2\frac{\sin A\sin C}{\sin(A+C)} = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)}$$

HALF ANGLE FORMULAE

If $2s$ shows the perimeter of a triangle ABC then,

$$2s = a+b+c$$

1. **Formulae for** $\sin\frac{A}{2},\quad \sin\frac{B}{2},\quad \sin\frac{C}{2}$

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ca}}$ (iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

2. **Formulae for** $\cos\frac{A}{2},\quad \cos\frac{B}{2},\quad \cos\frac{C}{2}$

(i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

3. **Formulae for** $\tan\frac{A}{2},\quad \tan\frac{B}{2},\quad \tan\frac{C}{2}$

(i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{(s-c)(s-a)}{\Delta}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{(s-a)(s-b)}{\Delta}$ (iv) $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \frac{s(s-a)}{\Delta}$

(v) $\cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \frac{s(s-b)}{\Delta}$ (vi) $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s(s-c)}{\Delta}$.

Note : $(a+b-c)(b+c-a)(c+a-b) = a^2b+b^2a+a^2c+ac^2+b^2c+bc^2-a^3-b^3-c^3$

OBLIQUE TRIANGLE

The triangle which are not right-angled is called **oblique triangle**. We can solve a triangle if we know three of its parts at least one of which is a side. Different cases are as follows.

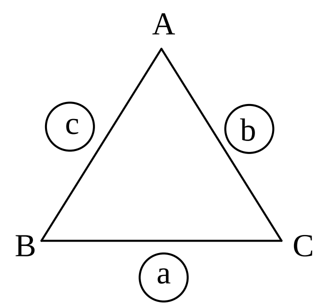
Case I. The three sides are given.
Case II. Two sides and included angles are given.
Case III. Two sides and the angle opposite to one of them are given.
Case IV. One side and two angles are given.

CASE I. Given the three sides, to solve the triangle.

Proof : Let *ABC* be the triangle in which all the tree sides *a*, *b*, *c* are given. The three angle *A*, *B*, *C* are to be determined.

Since, $2s=a+b+c$

$\therefore$ values of $s,\ s-a,\ s-b,\ s-c$ are known.



(i) *To find A*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$\therefore$ $\log\tan\frac{A}{2}=\frac{1}{2}\left[\log(s-b)+\log(s-c)-\log s-\log(s-a)\right]$

$\therefore$ $\log\tan\frac{A}{2}$ and $\therefore$ $\frac{A}{2}$ can be determined with the help of tables.

$\therefore$ A is determined.

(ii) *To find B*

$$\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$\therefore \quad \log \tan \frac{B}{2} = \frac{1}{2}\left[\log(s-a) + \log(s-c) - \log s - \log(s-b)\right]$

$\therefore \quad \log \tan \frac{B}{2}$ and $\quad \therefore \ \frac{B}{2}$ can be determined with the help of tables.

$\therefore \quad B$ is determined.

(iii) *To find C*

$A + B + C = 180°$

$\therefore \quad C = 180° - \overline{A+B}$, is determined.

Thus, A, B, C being known, the triangle is solve.

CASE II : Given two sides and the included angle ; to solve the triangle.

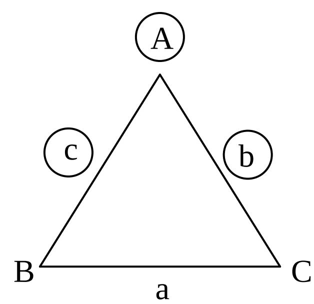
Proof : Let *ABC* be a triangle, in which sides *b*, *c* (*b* > *c*) and the included angle *A* are given. The side *a* and angles *B*, *C* are to be determined.

(i) *To find B and C.*

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad [\textit{Napier's Analogy}]$$

$$= \frac{b-c}{b+c} \tan \frac{B+C}{2} \qquad \left[\because \cot \frac{A}{B} = \cot\left(90° - \frac{B+C}{2}\right) = \tan \frac{B+C}{2}\right]$$

$\therefore \quad \log \tan \frac{B-C}{2} = \log(b-c) + \log \tan \frac{B+C}{2} - \log(b+c)$



$\therefore \quad \log \tan \frac{B-C}{2}$ and

$\therefore \quad \frac{B-C}{2}$ can be obtained with the help of tables.

$\therefore \quad \frac{B-C}{2}$ is known. …(1)

Also $\frac{B+C}{2} = 90° - \frac{A}{2} \qquad [\because A + B + C = 180°]$ …(2)

$\therefore$ from (1) and (2), by addition and subtraction *B* and *C* are known.

(ii) *To find a*

since $\qquad \frac{a}{\sin A} = \frac{b}{\sin B} \qquad$ [*Sine formula*]

$\therefore \qquad a = \frac{b \sin A}{\sin B}$

$\therefore \qquad \log a = \log b + \log \sin A - \log \sin B$

$\therefore \quad \log a$ and $\quad \therefore \ a$ can be determined with the help of the table.

Thus, *B*, *C* and *a* are known, the triangle is solved.

Note: If $C > B$, then use the formula

$$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2}.$$

CASE III : Given one sides and two angles ; to solve the triangle.

Proof : Let *ABC* be a triangle in which '*a*' be the given side and *B*, *C* be the given angles. Sides b, c and angle A are to be determined.

(i) *To find A*

$A + B + C = 180°$

$\therefore \quad A = 180° - (B + C)$

$\therefore \quad A$ is known.

(ii) *To find b*

Since $\quad \dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$$b = \frac{a \sin B}{\sin A} \qquad \text{[\textit{Sine formula}]}$$

$\therefore \quad \log b = \log a + \log \sin B - \log \sin A$

$\therefore \quad$ With the help of tables, $\log b$ and therefore, b is determined.

(iii) *To find c*

Again $\quad \dfrac{c}{\sin C} = \dfrac{a}{\sin A} \qquad$ [*Sine formula*]

$$\therefore \quad c = \frac{a \sin C}{\sin A}$$

$\therefore \quad \log c = \log a + \log \sin C - \log \sin A$

$\therefore \quad$ With the help of the tables, $\log c$ and therefore c is determined.

Thus, A, b, c being known, the triangle is solved.

CASE IV : When two sides and an angle opposite to one of them is given. (Ambiguous case)

Let the two sides say a and b and an angle A opposite to a be given.

Here we use $a / \sin A = b / \sin B$.

$\therefore \quad \sin B = b \sin A / a \qquad \ldots(1)$

We calculate angle B from (1) and then angle C is obtained by using

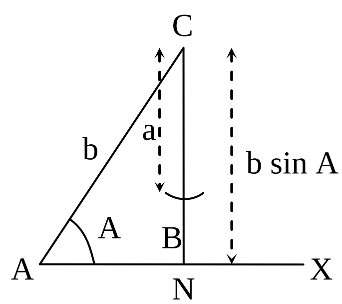
$$\angle C = 180° - (\angle A + \angle B).$$

Also, to find side c, we use

$$a / \sin A = c / \sin C$$

$\therefore \quad c = (a \sin C) / \sin A \qquad \ldots(2)$

From relation (1), the following possibilities will arise :



Case I : *When A is an acute angle.*

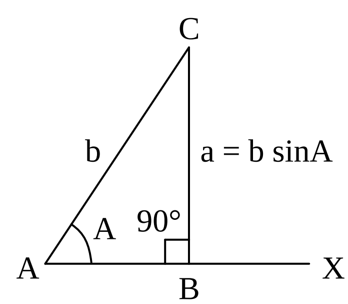
(a) If $a < b \sin A$, there is no triangle. When $a < b \sin A$, from (1), $\sin B > 1$, which is impossible. Hence no triangle is possible in this case.

From the following fig., if

$$AC = b \;;\; \angle CAX = A,$$

then perpendicular $CN = b \sin A$. Now taking c as centre, if we draw an arc of radius a then if the line AX and hence no triangle ABC can be constructed in this case.

(b) If $a = b \sin A$, then only one triangle is possible which is right angled at B.



When $a = b \sin A$, then from (1),

$$\sin B = 1, \qquad \therefore \angle B = 90°$$

From fig. it is clear that

$$CB = a = b \sin A.$$

Thus, in this case, only one triangle is possible which is right angled at B.

(c) If $a > b\sin A$, then further three possibilities will arise :

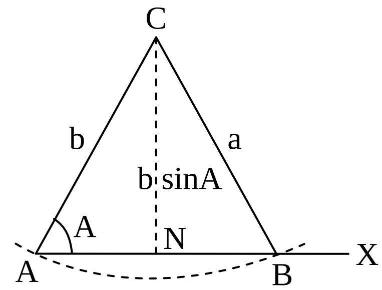
(i) $a = b$. In this case, from (1),

$$\sin B = \sin A$$

$\therefore$ $B = A$ or $B = 180° - A$

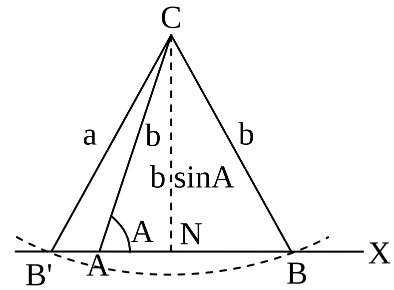
But $B = 180 - A \Rightarrow A + B - 180°$, which is not possible in a.

In this case we get $\angle A = \angle B$. Hence, if $b = a > b\sin A$ then only isosceles triangle ABC is possible in which $\angle A = \angle B$.



(ii) $a > b$. In the following fig., let $AC = b, \angle CAX = A$, and $a > b$, also $a > b\sin A$. Now taking C as centre, if we draw an arc of radius a, it will intersect AX at one point B and hence only one ΔABC is constructed. Also this arc will intersect XA produced at B' and $\Delta\ AB'C$ is also formed but this Δ is in admissible (because $\angle CAB$ is an obtuse angle in this triangle).

Hence, if $a > b$ and $a > b\sin A$, then only one triangle is possible.

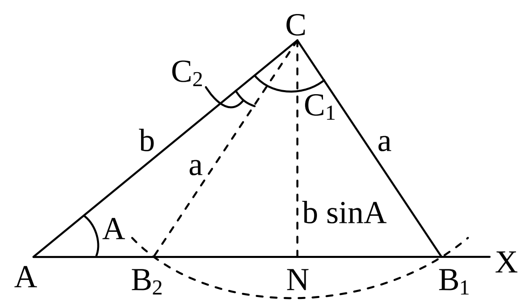


(iii) $b > a\left(i.e., b > a > b\sin A\right)$.

In the following fig., let

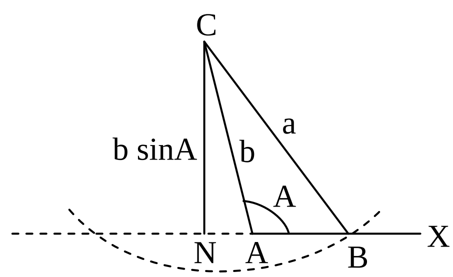
$$AC = b, \angle CAX = A.$$

Now taking C as centre, if we draw an arc of radius a, then it will intersect AX at two points B_1 and B_2. Thus two triangles AB_1C and AB_2C are formed.



Hence, if $b > a > b\sin A$, then there are two triangles.

Case II : *When A is an obtuse angle.*
In this case, there is only one triangle, if $a > b$.



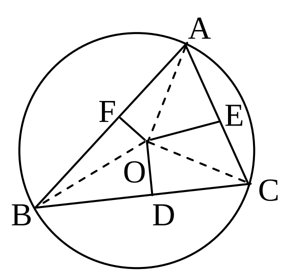
CIRCLE CONNECTED WITH TRIANGLE

1. **Circumcircle of a triangle its radius**

The circum-radius of a ΔABC is given by

(i) $\dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = R$

(ii) $R = \dfrac{abc}{4\Delta}\ \left[\Delta = area\, of\ \Delta ABC\right]$



2. **Inscribed circle or in-circle of a triangle and its radius**

The radius r of the inscribed circle of a triangle ABC is given by

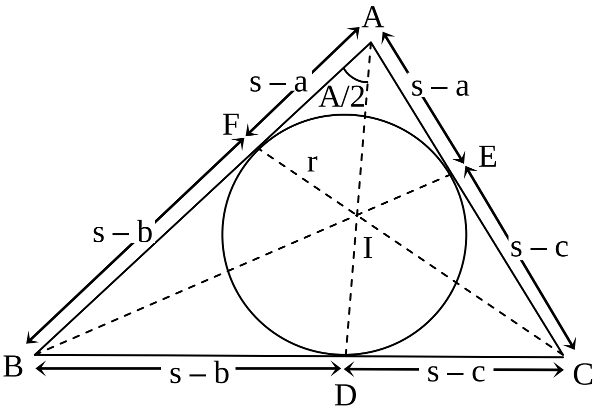
(i) $r = \dfrac{\Delta}{s}$

(ii) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iii) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

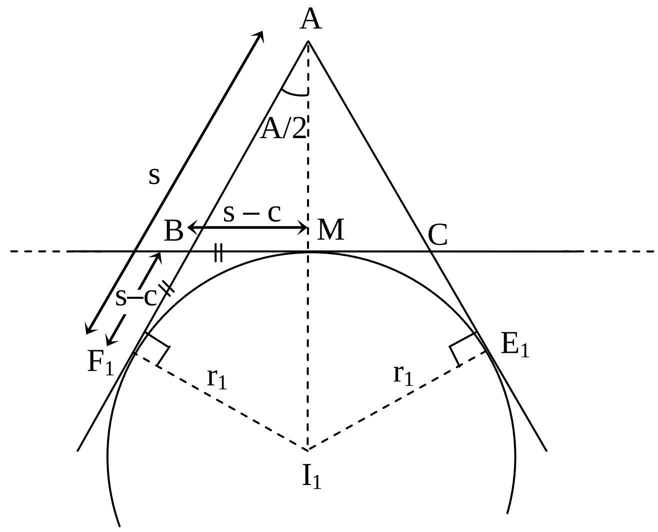
(iv) $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \dfrac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

(v) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$



3. **Escribed circle of a triangle and their radii**

In any ΔABC, we have



(i) $r_1 = \dfrac{\Delta}{s-a} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

$= s\tan\frac{A}{2} = (s-c)\cot\frac{B}{2}.$

$= (s-b)\cot\frac{C}{2} = \dfrac{a}{\tan\frac{B}{2} + \tan\frac{C}{2}}$

(ii) $r_2 = \dfrac{\Delta}{s-b} = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} = s\tan\frac{B}{2} = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2} = \dfrac{b}{\tan\frac{C}{2} + \tan\frac{A}{2}}$

(iii) $r_3 = \dfrac{\Delta}{s-c} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} = s\tan\frac{C}{2} = (s-b)\cot\frac{A}{2} = (s-a)\cot\frac{B}{2} = \dfrac{c}{\tan\frac{A}{2} + \tan\frac{B}{2}}$

(iv) $r_1 + r_2 + r_3 - r = 4R$

(v) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

(vi) $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

(vii) $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$

(viii) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

(ix) $\Delta = 2R^2\sin A.\sin B.\sin C = 4Rr\cos\frac{A}{2}.\cos\frac{B}{2}\cos\frac{C}{2}$

(x) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$; $r_2 = 4R\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}$, $r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

DISTANCE OF CIRCUMCENTRE (O) FROM THE ORTHOCENTRE (H), INCENTRE (I) AND EXCENTRES (I_1, I_2, I_3)

1. Distance between circumcentre (O) and orthocentre (H)

$$OH = R\sqrt{1-8\cos A\cos B\cos C}$$

2. Distance between circumcentre (O) and incentre (I)

$$OI = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R}\sqrt{R-2r}$$

3. (a) Distance between circumcentre (O) and excentre (I_1) of the escribed circle having opposite angle A

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_1}$$

(b) Distance between circucentre (O) and excentre (I_2) of the escribed circle having opposite angle B

$$OI_2 = R\sqrt{1+8\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_2}$$

(c) Distance between circumcentre (O) and excentre (I_3) of the escribed circle having opposite angle C

$$OI_3 = R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R}\sqrt{R+2r_3}$$

DISTANCE OF INCENTRE FROM THE VERTICES OF THE TRIANGLE

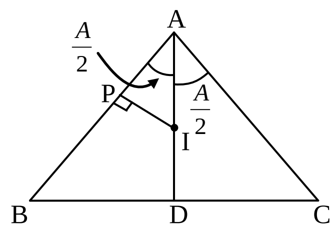
Let I be the In-centre. Let $IP. \perp AB.$ Clearly, $IP = r.\angle PAI = \frac{A}{2}$

From right angled triangle IPA,

$$\sin\frac{A}{2} = \frac{r}{AI} \Rightarrow AI = r\,\text{cosec}\frac{A}{2}$$

Similarly $BI = r\ \text{cosec}\frac{B}{2}$ and $CI = r\ \text{cosec}\frac{C}{2}$

Thus, $AI = r\ \text{cosec}\frac{A}{2},\ BI = r\ \text{cosec}\frac{B}{2}$ and $CI = r\ \text{cosec}\frac{C}{2}$.



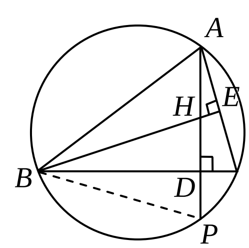
Note :

(i) The centroid of any triangle divides the join of circumcentre and orthocentre internally in the ratio 1 : 2.

(ii) If H is the orthocentre of ΔABC and AH produced meets BC at D and the circumcircle of ΔABC at P, then $HD = DP$.

$\because BD = c\cos B$ and $DP = BD\cot C$

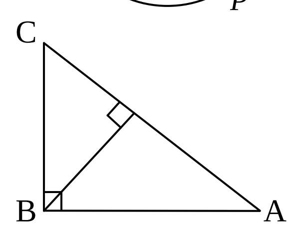
$\therefore DP = c\cos B.\cot C = 2R\cos B\cos C$



(iii) The orthocentre of an acute angled triangle is the incentre of the Pedal triangle.

(iv) The centre of the circum circle falls inside the triangle if triangle is acute angled but outside when it is obtuse angled. If the triangle is right angled the centre lies on mid-point of the hypotenuse.

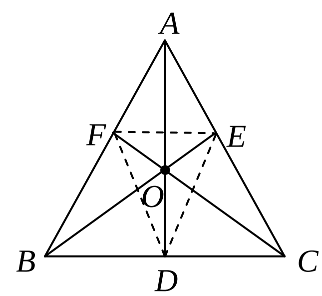
(v) The orthocentre falls inside the triangle if triangle is acute angled and outside when it is obtuse angled.
If the triangle is right angled the orthocentre (B) lies on the triangle.



PEDAL TRIANGLE

Let the perpendicular *AD*, *BE* and *CF* from the vertices *A*, *B* and *C* on the opposite sides *BC*, *CA* and *AB* of $\Delta\ ABC$ respectively, meet at *O*. Then *O* is the orthocentre of the ΔABC. The triangle *DEF* is called the pedal triangle of ΔABC.

Orthocentre of the triangle is the incentre of the pedal triangle. If *O* is the orthocentre and *DEF* the pedal triangle of the ΔABC, where *AD*, *BE*, *CF* are the perpendiculars drawn from *A*, *B*, *C* on the opposite side *BC*, *CA*, *AB* respectively, then



(i) $OA = 2R\cos A,\ OB = 2R\cos B$ and $OC = 2R\cos C$

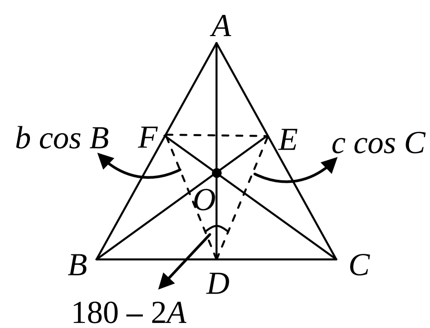
(ii) $OD = 2R\cos B\cos C,\ OE = 2R\cos C\cos A$

and $OF = 2R\cos A\cos B$

1. **Sides and angles of a pedal triangle**

The angles of pedal triangle *DEF* are : $180 - 2A, 180 - 2B,\ 180 - 2C$ and sides of pedal triangle are : $EF = a\cos A$ or $R\sin 2A$; $FD = b\cos B$ or $R\sin 2B$; $DE = c\cos C$ or $R\sin 2C$.

If given $\Delta\ ABC$ is obtuse, then angles are represented by $2A,\ 2B,\ 2C - 180°$ and the sides are $a\cos A,\ b\cos B,\ -c\cos C$.



2. **Area and circum-radius and in-radius of pedal triangle**

Area of pedal triangle $= \frac{1}{2}$ (Product of sides) $\times$ (sine of included angle)

$$\Delta = \frac{1}{2}R^2 \cdot \sin 2A \sin 2B \sin 2C$$

Circum-radius of pedal triangle $= \dfrac{EF}{2\sin FDE} = \dfrac{R\sin 2A}{2\sin(180° - 2A)} = \dfrac{R}{2}$

In-radius of pedal triangle $= \dfrac{area\ of\ \Delta\ DEF}{semi - perimeter\ of\ \Delta\ DEF} = \dfrac{\frac{1}{2}R^2\sin 2A.\sin 2B\sin 2C}{2R\sin A.\sin B.\sin C}$

$= 2R\cos A.\cos B.\cos C$.

EX-CENTRAL TRIANGLE

Let ABC be triangle and *I* be the centre of incircle. Let I_1, I_2 and I_3 be the centres of the escribed circle which are opposite to A, B, C respectively then $I_1I_2I_3$ is called the Ex-central triangle of ΔABC.

$I_1I_2I_3$ is a triangle, thus the triangle ABC is the pedal triangle of its ex-central triangle $I_1I_2I_3$.

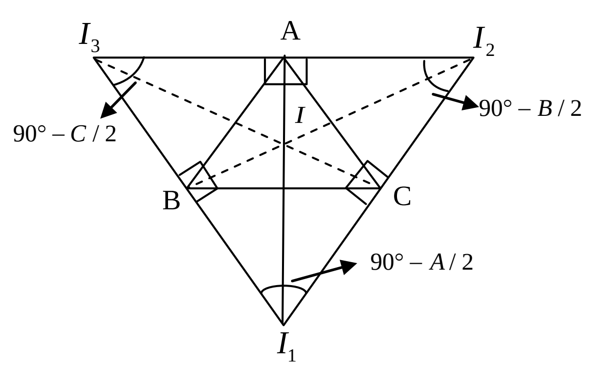
The angle of ex-central triangle $I_1I_2I_3$

are $90^0 - \dfrac{A}{2}, 90^0 - \dfrac{B}{2}, 90^0 - \dfrac{C}{2}$

$II_1 = 4R\sin\dfrac{A}{2},\quad II_2 = 4R\sin\dfrac{B}{2},\quad II_3 = 4R\sin\dfrac{C}{2}$

and sides are

$I_1I_3 = 4R\cos\dfrac{B}{2}; I_1I_2 = 4R\cos\dfrac{C}{2}; I_2I_3 = 4R\cos\dfrac{A}{2}$.



Area and circum-radius of the ex-central triangle

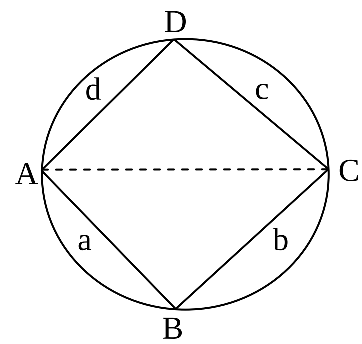
Area of triangle

$=\frac{1}{2}$(Product of two sides) $\times$(sine of included angles) $\Rightarrow \Delta=\frac{1}{2}\left(4R\cos\frac{B}{2}\right).\left(4R\cos\frac{C}{2}\right)\times\sin\left(90^0-\frac{A}{2}\right)$

$\Delta=8R^2\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$ Circum-radius $=\frac{I_2I_3}{2\sin I_2I_1I_3}=\frac{4R\cos\frac{A}{2}}{2\sin\left(90^0-\frac{A}{2}\right)}=2R.$

CYCLIC QUADRILATERAL

A quadrilateral ABCD is said to be cyclic quadrilateral if there exists a circle passing through all its four vertices A, B, C and D. Let a cyclic quadrilateral be such that]



$AB=a,\ BC=b,\ CD=c$ and $DA=d$.

Then $\angle B+\angle D=180°,\ \angle A+\angle C=180°$

Let $2s=a+b+c+d$,

Area of cyclic quadrilateral $=\frac{1}{2}(ab+cd)\sin B$

Also, area of cyclic quadrilateral $=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Where 2s = a + b + c + d and $\cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$.

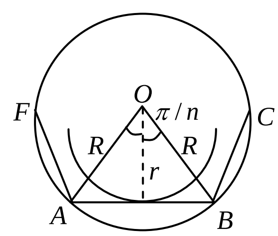
Circumradius of cyclic quadrilateral

Circum circle of quadrilateral ABCD is also the circumcircle of ΔABC. $R=\frac{1}{4\Delta}\sqrt{(ac+bd)(ad+bc)(ab+cd)}=\frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$.

REGULAR POLYGON

A regular polygon is a polygon which has all its sides equal and all its angles equal.

1. Each interior angle of a regular polygon of n sides is $\left(\frac{2n-4}{n}\right)\times$ right angles $=\left[\frac{2n-4}{n}\right]\times\frac{\pi}{2}$ radians.
2. The circle passing through all the vertices of a regular polygon is called its *circumscribed* circle.
   If a is the length of each side of a regular polygon of n sides, then the radius R of the circumscribed circle, is given by $R=\frac{a}{2}\text{cosec}\left(\frac{\pi}{n}\right)$.
3. The circle which can be inscribed within the regular polygon so as to touch all its sides is called its inscribed circle.



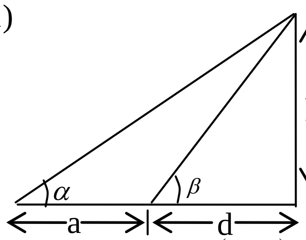
   Again if a is the length of each side of a regular polygon of n sides, then the radius r of the inscribed circle is given by $r=\frac{a}{2}\cot\left(\frac{\pi}{n}\right)$
4. The area of a regular polygon is given by $\Delta=n\times$area of triangle OAB

   $=\frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)$, (in terms of side)

   $=nr^2.\tan\left(\frac{\pi}{n}\right)$, (in terms of in-radius) $=\frac{n}{2}.R^2\sin\left(\frac{2\pi}{n}\right)$, (in terms of circum-radius).

SOME IMPORTANT RESULTS

(1)

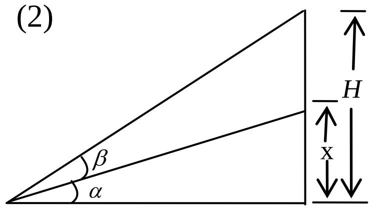


$a = h(\cot\alpha - \cot\beta) = \dfrac{h\sin(\beta-\alpha)}{\sin\alpha.\sin\beta}$

$\therefore h = \sin\alpha + \sin\beta \operatorname{cosec}(\beta\text{-}\alpha)$ *and*

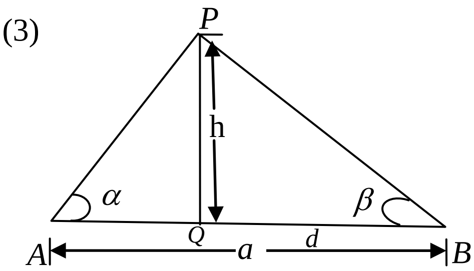
$d = h\cot\beta = a\sin\alpha.\cos\beta.\operatorname{cosec}(\beta-\alpha)$

(2)



$$H = x\cot\alpha\tan(\alpha+\beta)$$

(3)

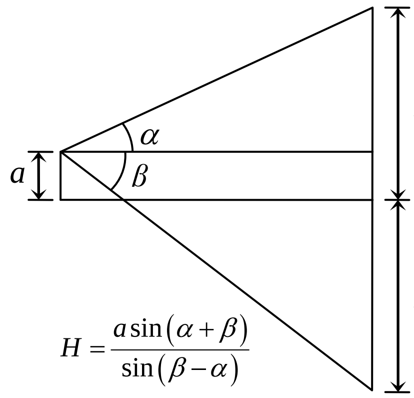


$a = h(\cot\alpha + \cot\beta)$, where by

$h = a\sin\alpha\sin\beta.\operatorname{cosec}(\alpha+\beta)$ and

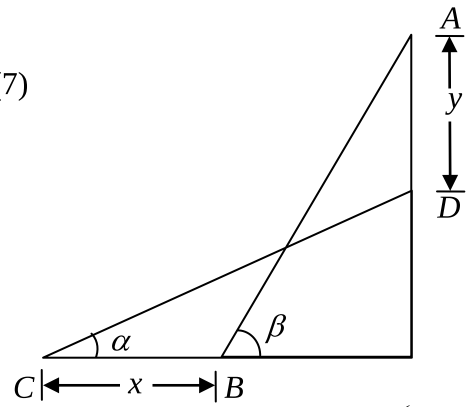
$d = h\cot\beta = a\sin\alpha.\cos\beta.\operatorname{cosec}(\alpha+\beta)$

(6)



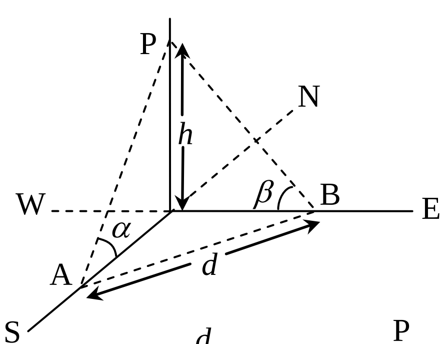
$$H = \frac{a\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$$

(7)



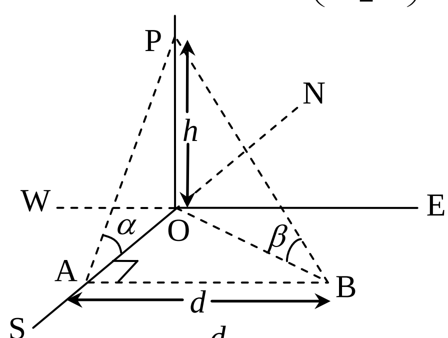
If $AB = CD$. Then, $x = y\tan\left(\dfrac{\alpha+\beta}{2}\right)$

(8)



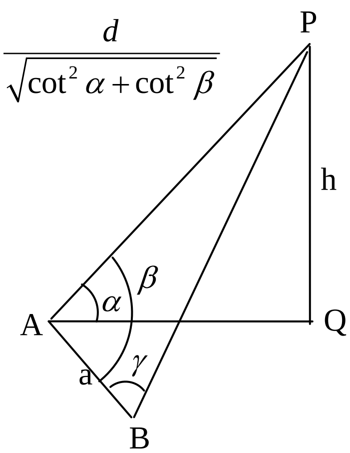
$$h = \frac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$$

(9)



$$h = \frac{d}{\sqrt{\cot^2\alpha - \cot^2\beta}}$$

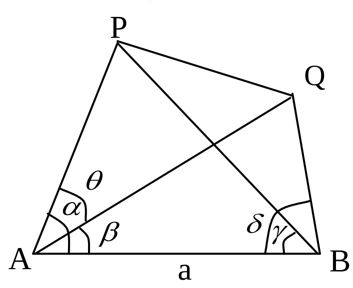
(10)



$$h = AP\sin\alpha = \alpha\sin\alpha.\sin\gamma.\operatorname{cosec}(\beta-\gamma)$$

If $AQ = d = AP\cos\alpha = \alpha\cos\alpha.\sin\gamma.\operatorname{cosec}(\beta-\gamma)$

(11)



$AP = a\sin\gamma.\operatorname{cosec}(\alpha-\gamma)$,

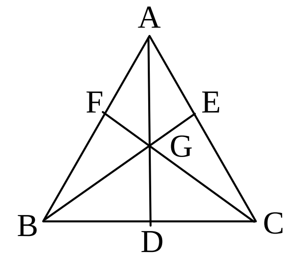
$AQ = a\sin\delta.\operatorname{cosec}(\beta-\delta)$

and apply,

$PQ^2 = AP^2 + AQ^2 - 2AP.AQ\cos\theta$

TIPS & TRICKS

The length of the medians AD, BE, CF of ΔABC are given by



$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \qquad = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A}$

$BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} = \frac{1}{2}\sqrt{c^2 + a^2 + 2ca.\cos B}$

$CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} = \frac{1}{2}\sqrt{a^2 + b^2 + 2ab.\cos C}$

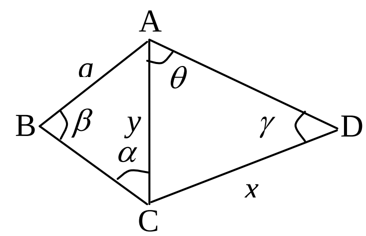
The distance between the circumcentre O and the in centre

$I\ of\ \Delta ABC\ given\,by\, OI = R\sqrt{1 - 8\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}}$

If I_1 is the centre of the escribed circle opposite to the

Angle A, then $OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}, \qquad OI_3 = R\sqrt{1 + 8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$

Similarly $OI_2 = R\sqrt{1 + 8\cos A/2.\sin B/2.\cos C/2}$



In the application of sine rule, the following point be noted. We are given one side a and some other side x is to be found. Both these are in different triangles.

We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y .

Thus, we will get unknown side x in terms of a. In the adjoining figure a is known side of ΔABC and x is unknown is side of triangle ACD. The common side of these triangle is $AC = y$ (say).

Now apply sine rule

$\therefore \qquad \frac{a}{\sin\alpha} = \frac{y}{\sin\beta} \quad ...(\text{i}) \qquad \text{and} \qquad \frac{x}{\sin\theta} = \frac{y}{\sin\gamma} \quad ...(\text{ii})$

Dividing (ii) by (i) we get,

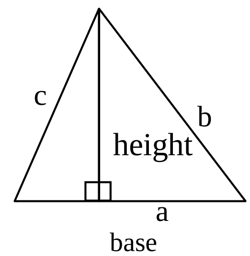
$\frac{x\sin\alpha}{a\sin\theta} = \frac{\sin\beta}{\sin\gamma}; \qquad \therefore\ x = \frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}$.

Mensuration

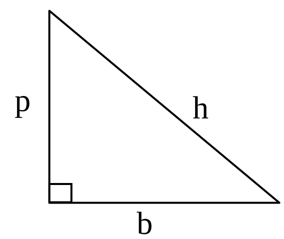
Chapter 15

AREA AND PERIMETER

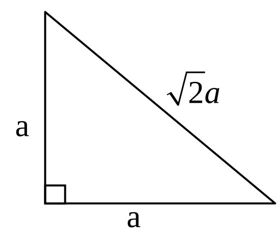
1. **Triangle**
   (a) Perimeter (2s) = a + b + c
   (b) Area $= \sqrt{s(s-a)(s-b)(s-c)}$
   (c) Area $= \frac{1}{2} \times base \times height$ where a, b, c are sides of the triangle and S is semi-perimeter, $S = \frac{a+b+c}{2}$



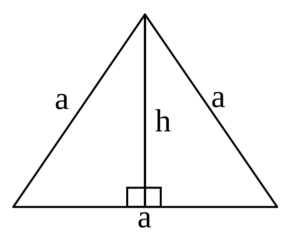
2. **Right-angled triangle**
   (a) Perimeter = b + p + h
   (b) Area = $\frac{1}{2} \times b \times p$
   (c) Hypotenuse (h) = $\sqrt{b^2 + p^2}$
   where b = base, p = perpendicular, h = hypotenuse



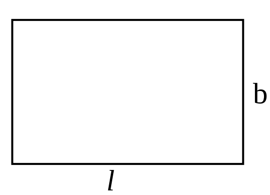
3. **Right-angle Isosceles triangle**
   (a) Hypotenuse = $\sqrt{a^2 + a^2} = \sqrt{2}a$
   (b) Perimeter = $2a + \sqrt{2}a$
   (c) Area = $\frac{1}{2} \times base \times height = \frac{1}{2} \times a \times a = \frac{1}{2}a^2$



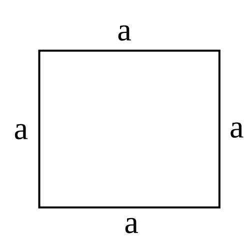
4. **Equilateral triangle**
   (a) Perimeter = 3a
   (b) Height $= \frac{\sqrt{3}}{2}a$
   (c) Area $= \frac{\sqrt{3}}{4}a^2$



5. **Rectangle**
   (a) Perimeter = 2 (l+ b)
   (b) Diagonal = $\sqrt{l^2 + b^2}$
   (c) Area $= l \times b$

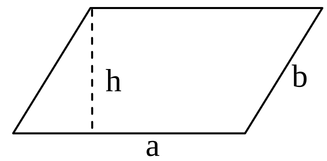


6. **Square**
   (a) Perimeter = 4a
   (b) Diagonal (d) $= \sqrt{2}a$



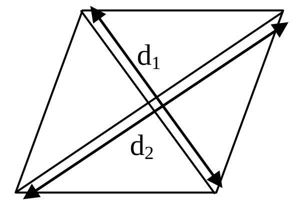
(c) Area = $a^2 = \frac{1}{2}d^2$

7. **Parallelogram**



(a) Perimeter = 2 (a + b)
(b) Area = Base × Height

8. **Rhombus**

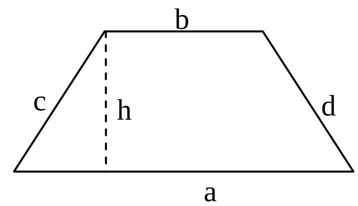


(a) Side = $\frac{1}{2}\sqrt{d_1^2 + d_2^2}$

(b) Perimeter = $2\sqrt{d_1^2 + d_2^2}$

(c) Area = $\frac{1}{2}d_1 d_2$

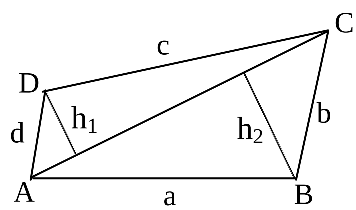
9. **Trapezium**



(a) Area = $\frac{1}{2}(a+b)\times h$

(b) Perimeter = a + b + c + d

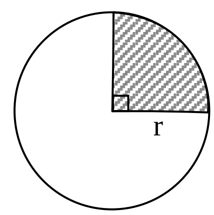
10. **Quadrilateral**



(a) Area = $\frac{1}{2}\times AC\times(h_1+h_2)$

(b) Perimeter = a + b + c + d

11. **Circle**



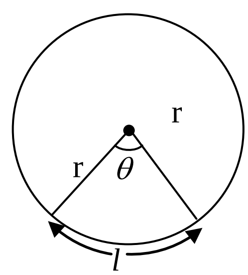
(a) Diameter (d) = 2r
(b) Circumference $= 2\pi r = \pi d$

(c) Area = $\pi r^2 = \frac{\pi d^2}{4}$

(d) Area of semi-circle = $\frac{\pi r^2}{2}$

(e) Area of quadrant = $\frac{\pi r^2}{4}$

12. **Sector**



(a) Area of sector (A) = $\frac{\theta}{360}\times\pi r^2 = \frac{1}{2}lr$

(b) Length of arc (l) = $\frac{\theta}{360}\times 2\pi r$

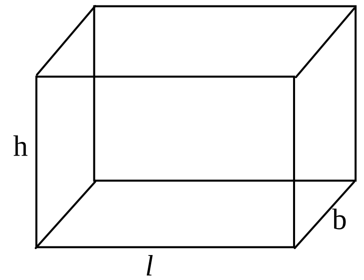
13. **Regular Polygon**

(a) Area = $\frac{1}{2}$ number of sides × radius of the inscribed circle

(b) Vertex angle of a regular polygon $(\theta) = \left(\frac{n-2}{n}\right)\times 180°$

14. **Volume and Surface Area**

(i) **Cuboid**



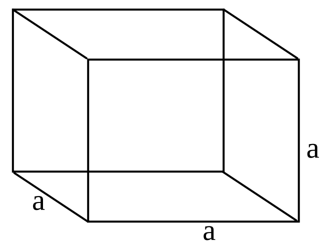
(a) Volume = $l \times b \times h$

(b) Surface Area = $2(lb + bh + hl)$

(c) Diagonal = $\sqrt{l^2 + b^2 + h^2}$

(d) Area of four walls = $2(l + b) \times h$

(ii) **Cube**

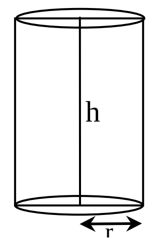


(a) Volume = a^3

(b) Surface Area = $6a^2$

(c) Diagonal = $\sqrt{3}a$

(iii) **Right Circular Cylinder**



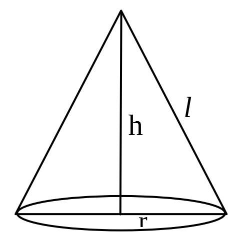
(a) Volume = $\pi r^2 h$

(b) Curved Surface Area $= 2\pi rh$

(c) Total Surface Area = $2\pi r(r + h)$

(d) Area of each end or base area = πr^2

(iv) **Right Circular Cone**



(a) Volume $= \frac{1}{3}\pi r^2 h$

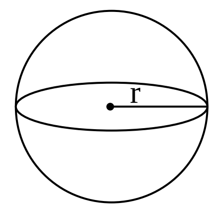
(b) Slant height = $\sqrt{r^2 + h^2}$

(c) Curved Surface Area = πrl

(d) Base Area = πr^2

(e) Total Surface Area = $\pi r(r + l)$

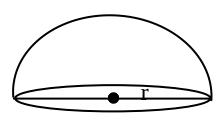
(v) **Sphere**



(a) Volume = $\frac{4}{3}\pi r^3$

(b) Total Surface Area = $4\pi r^2$

(vi) **Hemisphere**

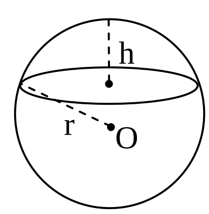


(a) Volume = $\frac{2}{3}\pi r^3$

(b) Total surface area = $3\pi r^2$

(c) Curved Surface Area = $2\pi r^2$

(d) **Spherical Gap of Radius '*r*' and Height '*h*'**



Volume $= 1/3\pi h^2(3r - h)$

Surface area $= 2\pi rh$

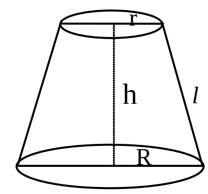
(vii) **Frustum of Cone**

(a) Slant height (l) $= \sqrt{h^2 + (R-r)^2}$

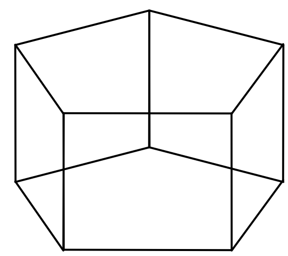
(b) Volume (V) $= \frac{\pi h}{3}\left(R^2 + r^2 + Rr\right)$

(c) Curved Surface Area $= \pi\left(R + r\right)l$

(d) Total Surface Area = $\pi\left[\left(R^2 + r^2\right) + l\left(R + r\right)\right]$



(viii) **Prism and Pyramid**



A. **Prism**

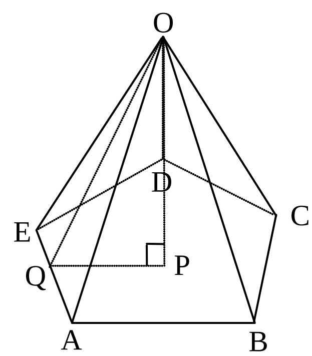
(a) Volume of a Right Prism = Area of the base $\times$ Height

(b) Lateral Surface Area = Perimeter of the Base $\times$ Height

B. **Pyramid**

(a) Volume of a Pyramid = $\frac{1}{3} \times$ Area of the base $\times$ height

(b) The whole surface area of a Pyramid is the sum of the areas of the base and the lateral surface areas.



Function Chapter 16

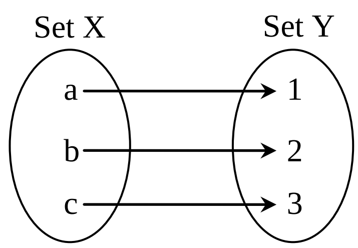
DEFINITION OF FUNCTION

Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y''. Let the correspondence be '*f*'' then mathematically we write $f: X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

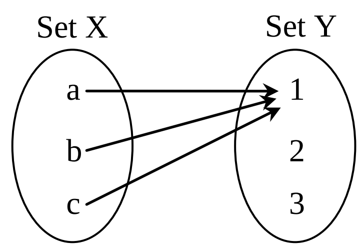
It is important to note that

(i) A mapping $f: X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there may be few elements in set Y which are not the images of any element in set X.

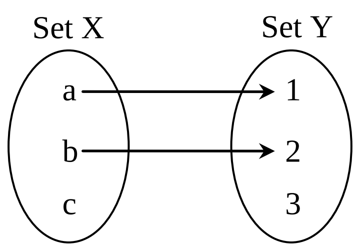
(ii) Every element in set X should have one and only one image. That means it is impossible to have more than one image for a specific element in set X. Functions can not be multi-values (A mapping that is multi-valued is called a relation from X and Y) *e.g.*



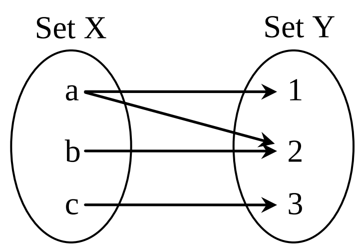
Function



Function



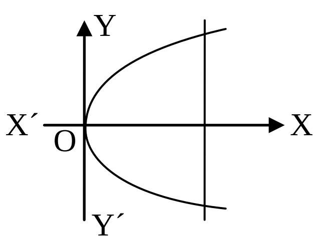
Not Function



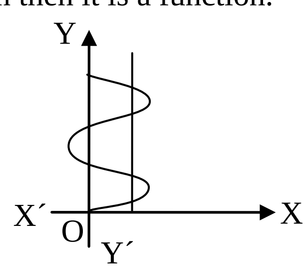
Not Function

TESTING FOR A FUNCTION BY VERTICAL LINE TEST

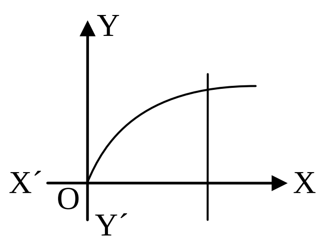
A relation $f: R \to R$ is a function or not it can be checked by a graph of the relation. If we draw a vertical line within the domain if any of these lines cuts the given curve at more than one point within the codomain then the given relation is not a function and when this vertical line means line parallel to Y-axis cuts the curve at only one point within the codomain then it is a function.



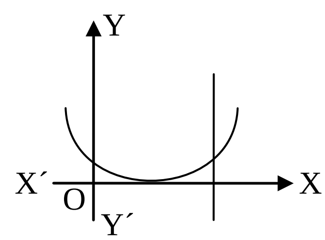
(i)



(ii)



(iii)



(iv)

Figure (i) and (ii) are representing a graph while figure (iii) and (iv) are representing a function.

NUMBER OF FUNCTIONS

Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

VALUE OF THE FUNCTION

If $y = f(x)$ is an function then to find its values at some value of *x*, say $x = a$, we directly substitute $x = a$ in its given rule $f(x)$ and it is denoted by $f(a)$.

e.g., If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$ etc.

DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

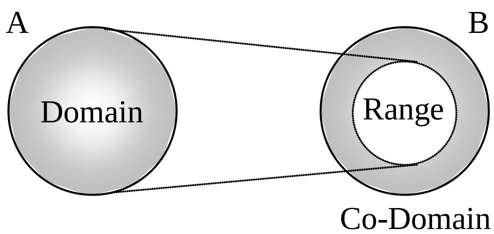
If a function *f* is defined from a set *A* to set *B* then for $f : A \to B$ set *A* is called the domain of function *f* and set *B* is called the co-domain of function *f*. The set of all *f*-images of the elements of *A* is called the range of function *f*.

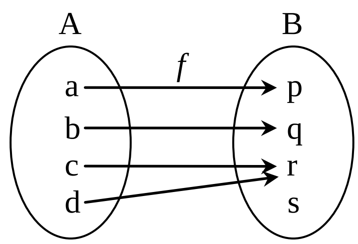
If nothing is specified about domain and codomain then

Domain = All possible values of *x* for which $f(x)$ exists.

And co-domain is taken to be *R*

Range = All possible values of $f(x)$, $\forall\, x$ in D_f.





Domain $= \{a, b, c, d\} = A$

Co- Domain $= \{p, q, r, s\} = B$

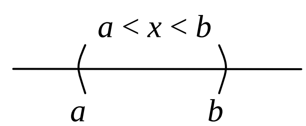
Range $= \{p, q, r\}$

INTERVALS

There are four types of interval.

1. **Open interval** : Let a and b be two real numbers such that $a < b$, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by $]a, b[$ or (a, b).

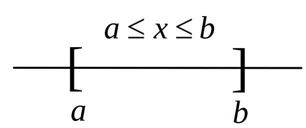
   Thus, $]a, b[$ or $(a, b) = \{x \in R : a < x < b\}$.



Open interval

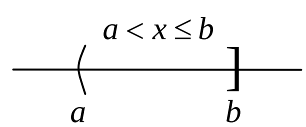
2. **Closed interval :** Let a and b be two real numbers such that $a < b$, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by $[a, b]$.

   Thus, $[a, b] = \{x \in R : a \le x \le b\}$.



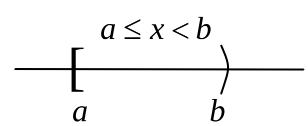
Closed interval

3. **Open-Closed interval :** It is denoted by $]a, b]$ or $(a, b]$ and $(a, b] = \{x \in R : a < x \le b\}$.



Open closed interval

4. **Closed-Open interval :** It is denoted by $[a, b[$ or $[a, b)$ and $[a, b[$ or $[a, b) = \{x \in R : a \le x < b\}$.



Closed open interval

ALGEBRA OF FUNCTIONS

1. **Scalar multiplication of a function :** $(c\, f)(x) = c\, f(x)$**,** where *c* is a scalar.
2. **Addition/subtraction of functions**

$$(f \pm g)(x) = f(x) \pm g(x).$$

3. **Multiplication of functions**

$$(fg)(x) = (gf)(x) = f(x)g(x).$$

4. **Division of functions**

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

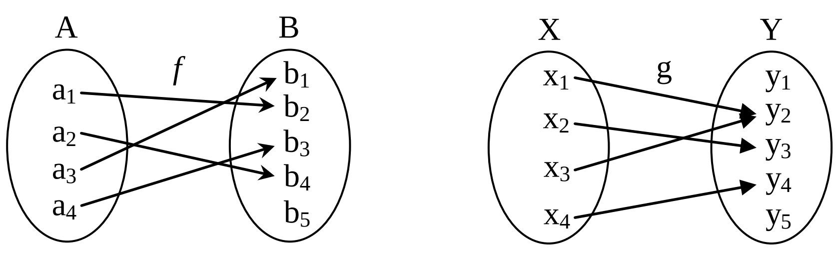
SOME IMPORTANT DEFINITIONS

1. **Real numbers :** Real numbers are those which are either rational or irrational. The set of real numbers is denoted by *R*.
2. **Related quantities :** When two quantities are such that the change in one is accompanied by the change in other, i.e., if the value of one quantity depends upon the other, then they are called related quantities.
3. **Variable :** A variable is a symbol which can assume any value out of a given set of values.
   (i) **Independent variable :** A variable which can take any arbitrary value, is called independent variable.
   (ii) **Dependent variable :** A variable whose value depends upon the independent variable is called dependent variable.
4. **Constant :** A constant is a symbol which does not change its value, i.e., retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc. There are two types of constant, absolute constant and arbitrary constant.
5. **Equal functions :** Two function f and g are said to be equal functions, if and only if
   (i) Domain of $f =$ Domain of g.
   (ii) Co-domain of $f =$ Co-domain of g.
   (iii) $f(x) = g(x)\ \forall\ x \in$ their common domain.
6. **Real valued function :** If R, be the set of real numbers and A, B are subsets of R, then the function $f : A \to B$ is called a real function or real-valued function.
   1. **One-one function (injection or injective function) :** A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of *A* have different images in *B*. Thus, $f : A \to B$ is one-one.

      $a \ne b \Rightarrow f(a) \ne f(b)$ for all $a, b \in A$

      $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

      e.g. let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams.



Clearly, $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function g.

(i) **Method to check the injectivity of a function**

Step I : Take two arbitrary elements x, y (say) in the domain of f.

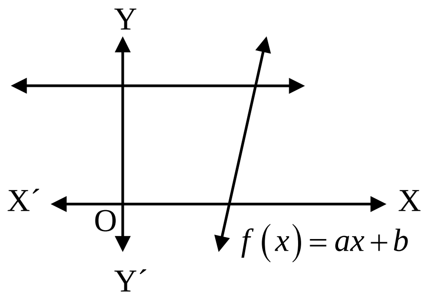
Step II : Put $f(x) = f(y)$.

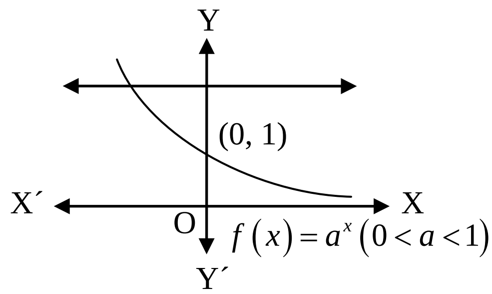
Step III : Solve $f(x) = f(y)$. If $f(x) = f(y) \Rightarrow x = y$ only, then $f : A \to B$ is a one-one function (or an injection). Otherwise not.

Note :

(a) If function is given in the form of ordered pairs and if any two ordered pairs do not have same second element then function is one-one.

(b) If the graph of the function $y = f(x)$ is given and we draw a line within the co- domain if each line parallel to x-axis cuts the given curve at maximum one point then function is one-one. e.g.





(c) All even functions are many one.

(d) All polynomials of even degree defined in R have at least one local maxima or minima and hence are may one in the domain R. Polynomials of odd degree may be one-one or many-one.

(e) If f is a rational function then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence we can write $f(x_1) - f(x_2) = (x_1 - x_2)\, g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-one else one-one.

(f) Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-one or many-one.

(ii) **Number of one-one functions (injections or injective functions) :** If A and B are finite sets having m and n elements respectively, then number of one-one function from A to B

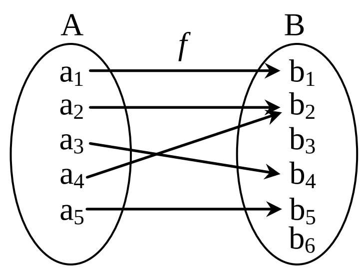
$$= \begin{cases} {}^nP_m, & if \;\; n \geq m \\ 0, & if \;\; n < m \end{cases}$$

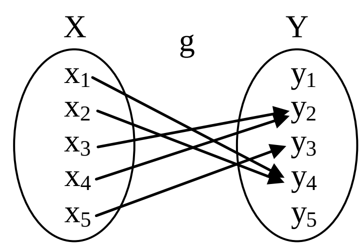
2. **Many-one function :** A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f : A \to B$ is a many-one function if there exist $x, \, y \in A$ such that $x \neq y$ but

$$f(x) = f(y).$$

In other words $f : A \to B$ is a many-one function if it is not a one-one function.

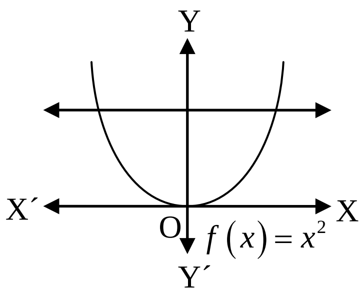


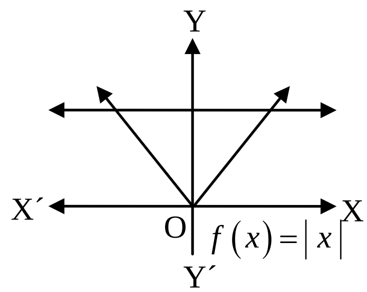


Note :

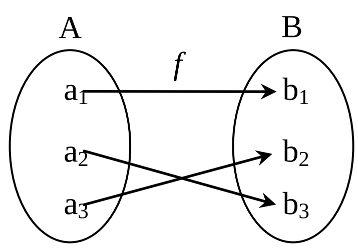
(a) If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

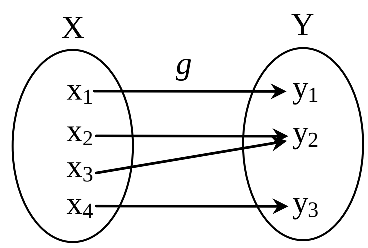
(b) If the graph of $y = f(x)$ is given and any line parallel to x-axis with in the codomain cuts the curve in its domain at more than one point then function is many-one.





3. **Onto function (surjection) or Surjective function :** A function $f : A \to B$ is onto if each element of B has its pre-image in A. Therefore, if $f^{-1}(y) \in A, \ \forall \ y \in B$ then function is onto. In other words. Range of $f =$ Co-domain of f e.g. The following arrow-diagram shows onto function.





Number of onto functions (surjections) : If A and B are two sets having n and r elements respectively such that $r \le n$, then number of onto functions from A to B is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n \,.........+ (-1)^{r-1}\, {}^rC_{r-1}.$$

4. **Into function :** A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A.

In other words, $f : A \to B$ is an into function if it is not an onto function e.g. The fig (i), arrow-diagram shows an into function.

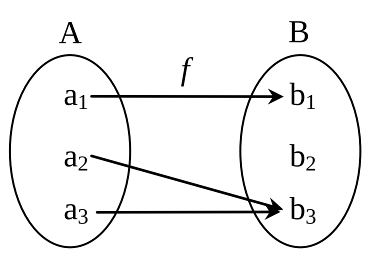


Fig (i)

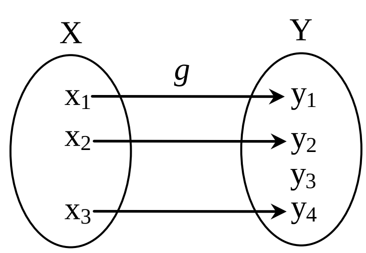


Fig (ii)

(i) **Method to find onto or into function**

(a) If Range = codomain, then f is onto. If range is a proper subset of codomain, then f is into.

(b) Solve $f(x) = y$ for x, say $x = g(y)$.

Now if $g(y)$ is defined for each $y \in$ codomain and $g(y) \in$ domain of f for all $y \in$ codomain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in codomain, then $f(x)$ is into.

Note :

(a) An into function can be made onto by redefining the codomain as the range of the original function.

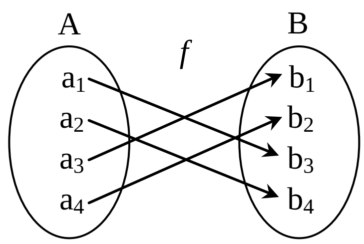
(b) Any polynomial function $f : R \to R$ is onto if degree is odd; into if degree of *f* is even.

5. **One-one onto function (bijection) or Bijective Function :** A function $f : A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f : A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x,\ y \in A$.

(ii) It is onto if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.



Clearly, f is a bijection since it is both injective as well as surjective.

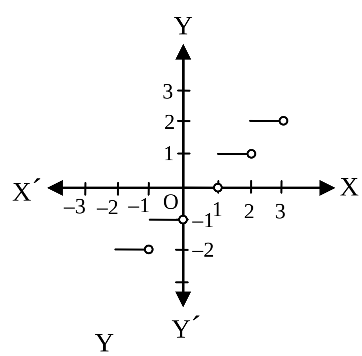
Number of one-one onto function (bijection) : If *A* and *B* are finite sets and $f : A \to B$ is a bijection, then *A* and *B* have the same number of elements. If *A* has *n* elements, then the number of bijection from *A* to *B* is the total number of arrangements of *n* items taken all at a time *i.e.*, $n!$.

6. **Absolute value :** The absolute value of a number x, denoted by $|x|$, is a number that satisfies the conditions

$$|x| = \begin{cases} -x \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ x \text{ if } x > 0 \end{cases}$$. We also define $|x|$ as follows,

$|x| = \text{maximum } \{x, -x\}$ or $|x| = \sqrt{x^2}$.

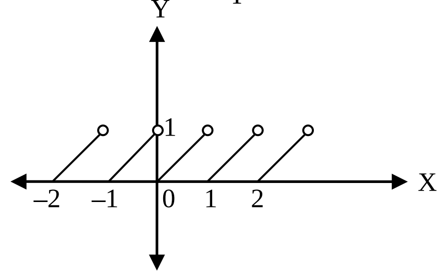
7. **Greatest integer function :** Let $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x. The domain is *R* and the range is *I*. *e.g.*, $[1.1] = 1,\ [2.2] = 2,\ [-0.9] = -1,\ [-2.1] = -3$ etc. The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function.



8. **Fractional Part Function :** We know that $x \geq [x]$.The difference between the number 'x' and it's integral value '$[x]$' is called the fractional part of x and is symbolically denoted as $\{x\}$.

Thus, $\{x\} = x - [x]$ *e.g.*, if $x = 4.92$ then $[x] = 4$ and $\{x\} = 0.92$.

Fractional part of any number is always non-negative and less than one.



9. **Least Integer Function :** $y = (x)$ indicates the integral part of x which is nearest and greatest integer to x.

Properties of greatest integral function :

(i) $[x] \le x < [x]+1$ and $x-1 < [x] \le x,\ 0 \le x-[x] < 1$.

(ii) $[x+m] = [x]+m$ if m is an integer.

(iii) $[x]+[y] \le [x+y] \le [x]+[y]+1$.

$[-x] = -[x]$ if $x \in I$

$[-x] = -[x]-1$ if $x \notin I$

(iv) $[x]+[-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

(v) If $[x] > n \Rightarrow x \ge n+1,\ n \in$ integer

(vi) If $[x] < n \Rightarrow x < n,\ n \in$ integer

(vii) $[x+y] = [x]+[y+x-[x]]$ for all $x,\ y \in R$

(viii) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if m is a positive integer.

(ix) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+....+\left[x+\frac{n-1}{n}\right] = [nx],\ n \in N$

(x) If $[\phi(x)] \ge I$, then $\phi(x) \ge I$.

(xi) If $[\phi(x)] \le I$, then $\phi(x) < I+1$.

(xii) $x-[x]$ is the fractional part of x.

(xiii) $-[-x]$ is the least integer $\ge x$.

(xiv) $[x+0.5]$ is the nearest integer to x. If two integers are equally near to $x.[x+0.5]$ denotes the nearest to x.

(xv) If n and a are positive integers, $[n/a]$ is the number of integers among 1, 2, ..., n that are divisible by a.

EXPONENT OF PRIME *p* in *n*!

Exponent of a prime p in $n!$ is denoted by $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+...+\left[\frac{n}{p^k}\right].$$

where $p^k < n < p^{k+1}$ and $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to $\frac{n}{p}$.

For example, exponent of 3 in $(100)!$ is $E_3(100!) = \left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]$

10. **Algebraic functions :** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, –, ×, and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}}+5x$ (ii) $\frac{\sqrt{x+1}}{x-1},\ x \ne 1$ (iii) $3x^4-5x+7$

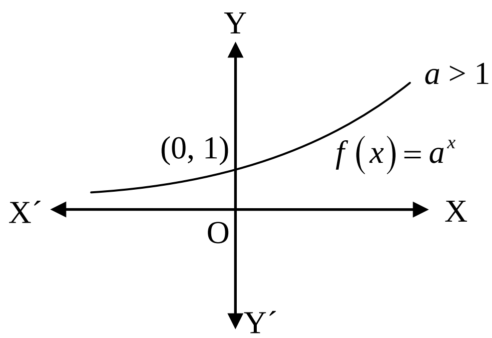
11. **Transcendental function :** A function which is not algebraic is called a transcendental function *e.g.*, trigonometric; inverse trigonometric, exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions :** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.

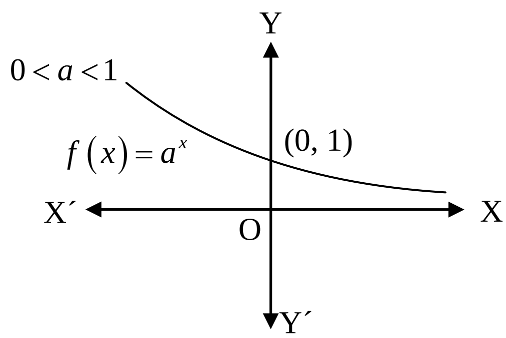
(ii) **Inverse trigonometric functions**

Function	Domain	Range	Definition of the function
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or R	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or R	$(0, \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\operatorname{cosec}^{-1} x$	$R-(-1, 1)$	$[-\pi/2, \pi/2]-\{0\}$	$y = \operatorname{cosec}^{-1} x \Leftrightarrow x = \operatorname{cosec} y$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\{\pi/2\}$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

(iii) **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ called exponential function. Its domain is R and range is $(0, \infty)$.

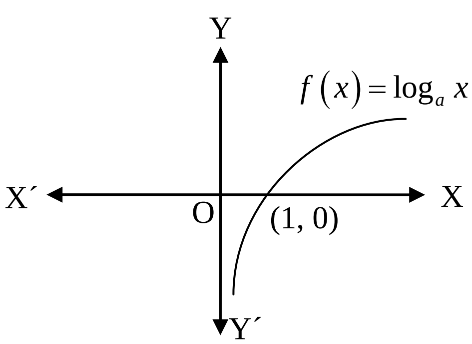


Graph of $f(x) = a^x$, when $a > 1$

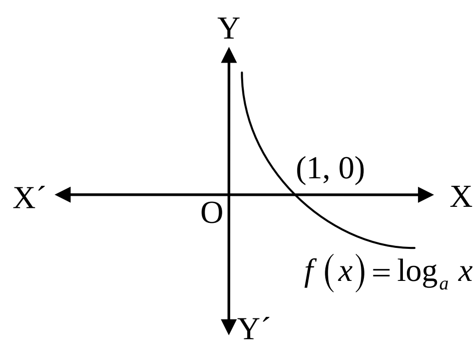


Graph of $f(x) = a^x$, when $0 < a < 1$

(iv) **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.



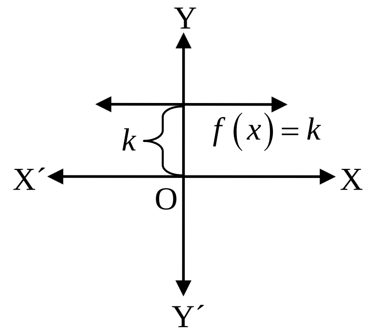
Graph of $f(x) = \log_a x$, when $a > 1$



Graph of $f(x) = \log_a x$, when $0 < a < 1$

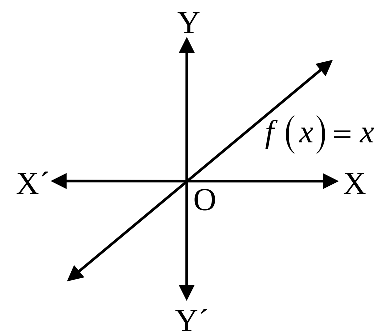
12. **Explicit and implicit functions :** A function is said to be explicit if it is expressed directly in terms of the independent variable. If the function is not expressed directly in terms of the independent variable or variables, then the function is said to be implicit e.g. $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a-x)^2 (b-y)^2$ are implicit functions.

13. **Constant function :** Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of f is the singleton set $\{k\}$. The graph of a constant function is a straight line parallel to *x*-axis as shown in figure and it is above or below the *x*-axis according as k is positive or negative. If $k = 0$, then the straight line coincides with *x*-axis.

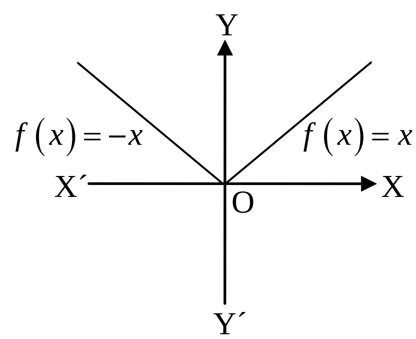


14. **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on *R*. Clearly, the domain and range of the identity function is *R*.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of *x*-axis.



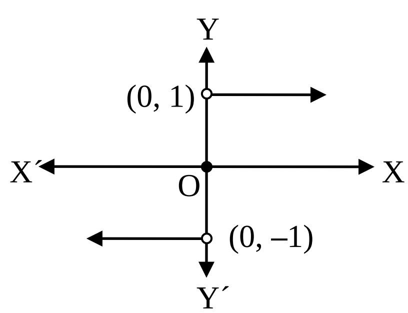
15. **Modulus function :** The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \ge 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set *R* of all real numbers and the range is the set of all non-negative real numbers.



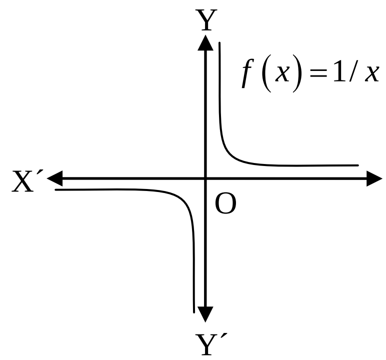
16. **Signum function :** The function defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases} \text{ or } f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function. The domain is R and the range is the set $\{-1, 0, 1\}$.



17. **Reciprocal function :** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R-\{0\}$ *i.e.*, the set of all non-zero real numbers. The graph is as shown.



18. **Power function :** A function $f: R \to R$ defined by, $f(x)=x^{\alpha}, \alpha \in R$ is called a power function.

EVEN AND ODD FUNCTION

1. **Even function :** If we put $(-x)$ in place of x in the given function and if $f(-x)=f(x), \forall x \in$ domain then function $f(x)$ is called even function

   e.g. $f(x)=e^{x}+e^{-x}$, $f(x)=x^2$, $f(x)=x\sin x$, $f(x)=\cos x$, $f(x)=x^2\cos x$ all are even functions.

2. **Odd function :** If we put $(-x)$ in place of x in the given function and if $f(-x)=-f(x), \forall\ x \in$ domain then $f(x)$ is called odd function. *e.g.*, $f(x)=e^{x}-e^{-x}$, $f(x)=\sin x$, $f(x)=x^3$, $f(x)=x\cos x$, $f(x)=x^2\sin x$ all are odd functions.

PROPERTIES OF EVEN AND ODD FUNCTION

1. Every function defined in symmetric interval D (i.e., $x \in D \Rightarrow -x \in D$) can be expressed as a sum of an even and an odd function.

$$f(x)=\left(\frac{f(x)+f(-x)}{2}\right)+\left(\frac{f(x)-f(-x)}{2}\right)$$

   Let $h(x)=\left(\frac{f(x)+f(-x)}{2}\right)$ and $g(x)=\left(\frac{f(x)-f(-x)}{2}\right)$. It can now easily be shown that $h(x)$ is even and $g(x)$ is odd.

2. The first derivative of an even differentiable function is an odd function and vice-versa.
3. If $x=0 \in$ domain of *f*, then for odd function $f(x)$ which is continuous at $x=0$, $f(0)=0$ i.e., if for a function, $f(0)\neq 0$, then that function can not be odd. It follows that for a differentiable even function $f'(0)=0$ i.e., if for a differentiable function $f'(0)\neq 0$ then the function *f* can not be even.
4. The graph of even function is always symmetric with respect to *y*-axis. The graph of odd function is always symmetric with respect to origin.
5. The product of two even functions is an even function.
6. The sum and difference of two even functions is an even function.
7. The sum and difference of two odd functions is an odd function.
8. The product of two odd functions is an even function.
9. The product of an even and an odd function is an odd function. It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function.

   e.g., $f(x)=x^2+x^3$, $f(x)=\log_e x$, $f(x)=e^{x}$.

10. The sum of even and odd function is neither even nor odd function.
11. Zero function $f(x)=0$ is the only function which is even and odd both.

PERIODIC FUNCTION

A function $f: X \to Y$ is said to be a periodic function if there exists a positive real number T such that $f(x+T)=f(x)$, for all $x \in X$. The least of all such positive numbers T is called the principal period or fundamental period of *f*. All periodic functions can be analysed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

To test for periodicity of the function we just need to show that $f(x+T)=f(x)$ for same $T(>0)$ independent of *x* whereas to find fundamental period we are required to find a least positive number *T* independent of *x* for which $f(x+T)=f(x)$ is satisfied for all *x*.

The following points are to be remembered :

If $f(x)$ is periodic with period T, then $af(x)+b$ where $a, b \in R(a \neq 0)$ is also periodic with period *T*.

1. If $f(x)$ is periodic with period T, then $f(ax+b)$ where $a, b \in R(a \neq 0)$ is also period with period $\frac{T}{|a|}$.
2. Let $f(x)$ has period $T_1 = m/n(m, n \in N \text{ and co-prime})$ and $g(x)$ has period $T_2 = r/s$ $(r, s \in N \text{ and co-prime})$ and T be the LCM of T_1 and T_2 i.e., $T = \frac{LCM \ of \ (m, r)}{HCF \ of \ (n, s)}$.

   Then T shall be the period of $f+g$ provided there does not exist a positive number $k(<T)$ for which $f(k+x)+g(k+x)=f(x)+g(x)$ else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.

Note :

LCM of p and q always exist if p/q is a rational quantity. If p/q is irrational then algebraic combination of *f* and *g* is non-periodic.

(a) $\sin^n x, \cos^n x, \text{cosec}^n x$ and $\sec^n x$ have period 2π if n is odd and π if n is even.

(b) $\tan^n x$ and $\cot^n x$ have period π whether n is odd or even.

(c) A constant function is periodic but does not have a well-defined period.

(d) If *g* is periodic then *fog* will always be a periodic function. Period of *fog* may or may not be the period of *g*.

(e) If f is periodic and g is strictly monotonic (other than linear) then *fog* is non-periodic.

COMPOSITE FUNCTION

If $f: A \to B$ and $g: B \to C$ are two function then the composite function of f and g.

$gof \ A \to C$ will be defined as $gof\ (x) = g[f(x)], \ \forall \ x \in A$

Properties of composition of function :

1. f is even, g is even $\Rightarrow$ *fog* even function.
2. f is odd, g is odd $\Rightarrow$ *fog* is odd function.
3. f is even, g is odd $\Rightarrow$ *fog* is even function.
4. f is odd, g is even $\Rightarrow$ *fog* is even function.
5. Composite of functions is not commutative *i.e.*, $fog \neq gof$.
6. Composite of functions is associative *i.e.*, $(fog)oh = fo(goh)$.

7. If $f: A \to B$ is bijection and $g: B \to A$ is inverse of f. Then $fog = I_B$ and $gof = I_A$. Where I_A and I_B are identity functions on the sets A and B respectively.
8. If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijection and $(gof)^{-1} = (f^{-1}og^{-1})$.
9. $fog \neq gof$ but if, $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ also, $(fog)(x) = (gof)(x) = (x)$.
10. $gof(x)$ is simply the g-image of $f(x)$, where $f(x)$ is f-image of elements $x \in A$.
11. Function gof will exist only when range of f is the subset of domain of g.
12. fog does not exist if range of g is not a subset of domain of f.
13. fog and gof may not be always defined.
14. If both f and g are one-one, then fog and gof are also one-one.
15. If both f and g are onto, then gof and fog are onto.

INVERSE OF A FUNCTION

If $f: X \to Y$ be a function defined by $y = f(x)$ such that *f* is both one-one and onto, then there exists a unique function $g: Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of *f* and denoted by f^{-1}. Also if *g* is the inverse of *f*, then *f* is the inverse of *g* and the two functions *f* and *g* are said to be inverses of each other.

The condition for existence of inverse of a function is that the function must be **one-one and onto**. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

Note : $fof^{-1}(x) = f^{-1}of(x) = x$ always and the graph of f and f^{-1} are symmetric about the line $y = x$.

Methods of Finding Inverse of a Function :

1. If you are asked to check whether the given function $y = f(x)$ is invertible, you need to check that $y = f(x)$ is one-one and onto.
2. If you are asked to find the inverse of a bijective function $f(x)$, you do the following :

   If f^{-1} be the inverse of *f*, then

   $f^{-1}of(x) = fof^{-1}(x) = x$ (always)

   Apply the formula of f on $f^{-1}(x)$ and use the above identity to solve for $f^{-1}(x)$.

Properties of Inverse function :

1. Inverse of a bijection is also a bijection function.
2. Inverse of a bijection is unique.
3. $(f^{-1})^{-1} = f$
4. If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$.
5. If $f: A \to B$ is bijection then $f^{-1}.B \to A$ is an inverse function of $f . f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here I_A is an identity function on set *A*, and I_B, is an identity function on set *B*.

Limit

Chapter 17

INDETERMINATE FORMS

If a function f assumes any one of the forms :

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0\times\infty,\ 1^{\infty},\ 0^0,\ \infty^0$$

at $x=a,$ then we say that f is indeterminate at $x=a.$

where 0 in every case denotes $\to 0$ and 1 denotes $\to 1,\ \infty$ denotes $\to\infty.$

NEIGHBOURHOOD OF A POINT

Let 'a' be a real number. Then for a positive real number $\delta>0,$ the open interval $(a-\delta,\ a+\delta)$ is called a neighbourhood of $a.$ $(a-\delta,\ a+\delta)-\{a\},$ i.e. is called a deleted neighbourhood of $a.$ The interval $(a-\delta,\ a)$ is called a left hand neighbourhood of a and $(a,\ a+\delta)$ is called a right hand neighbourhood of $a.$ If $x\in(a,\ a+\delta),$ we say that x approaches a from the right and we write $x\to a^+,$ etc.

LIMIT OF A FUNCTION

A real number l is called the limit of the function f defined in a deleted neighbourhood of 'a' as $x\to a$ if any given $\varepsilon>0,$ there exists $\delta>0$ such that

$|f(x)-l|<\varepsilon,$ whenever $0<|x-a|<\delta$ i.e. $l-\varepsilon<f(x)<l+\varepsilon,$ whenever $x\in(a-\delta)\cup(a+\delta)$ and we write $\lim\limits_{x\to a} f(x)=l$ or $f(x)\to l$ as $x\to a.$

Likewise, we can define $\lim\limits_{x\to a^-} f(x)=l$ and $\lim\limits_{x\to a^+} f(x)=l.$

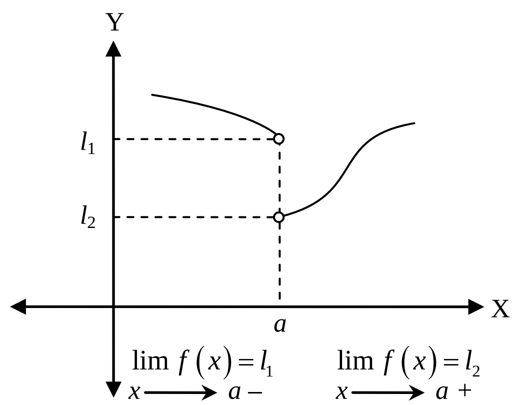
Note that $\lim\limits_{x\to a} f(x)=l$ iff $\lim\limits_{x\to a^-} f(x)=l=\lim\limits_{x\to a^+} f(x)$

ONE-SIDED LIMITS

The left-hand limit is written $\lim\limits_{x\to a-} f(x)=l$

If x approaches a from its left, that is it crosses over values of the form $a-h,\ h>0,$ the difference between $f(x)$ and l can be made arbitrarily small. More precisely, $\lim\limits_{x\to a-} f(x)=l$ if for each $\varepsilon>0,$ there exists a $\delta>0$ such that if $a-\delta<x<a,$ then $|f(x)-l|<\varepsilon.$

Similarly, we define right-hand limit and write it as $\lim\limits_{x\to a+} f(x)=l$



ALGEBRA OF LIMITS

If f and g are two real valued functions defined over the domain $D,$ we define functions $f\pm g,\ fg,\ f/g$ on D by

$$(f+g)(x)=f(x)\pm g(x),\quad (fg)(x)=f(x)g(x),$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}\qquad \text{(provided } g(x)\neq 0 \text{ for } x\in D\text{)}$$

We have :

(a) $\lim\limits_{x\to a}(f\pm g)(x)=\lim\limits_{x\to a} f(x)\pm\lim\limits_{x\to a} g(x)$

(b) $\lim\limits_{x\to a}(fg)(x) = \left(\lim\limits_{x\to a} f(x)\right)\left(\lim\limits_{x\to a} g(x)\right)$

(c) $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ (provided $\lim\limits_{x\to a} g(x) \neq 0$)

(d) $\lim\limits_{x\to a}(cf(x)) = c\lim\limits_{x\to a} f(x)$, where c is a constant.

METHODS OF EVALUATION OF LIMITS

We shall divide the problems of evaluation of limits in five categories.

1. **Algebraic limits :** Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim\limits_{x\to a} f(x)$ is known as an algebraic limit.
   - (i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.
   - (ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.
   - (iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalization the terms are factorised which on cancellation gives the result.
   - (iv) **Based on the form when $\mathbf{x\to\infty}$:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

2. **Trigonometric limits :** To evaluate trigonometric limit the following results are very important.

(i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{x}{\sin x}$

(ii) $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{x}{\tan x}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{x}{\sin^{-1} x}$

(iv) $\lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{x}{\tan^{-1} x}$

(v) $\lim\limits_{x\to 0}\cos x = 1$

(vi) $\lim\limits_{x\to a}\sin^{-1} x = \sin^{-1} a$, where $|a| \le 1$

(vii) $\lim\limits_{x\to a}\cos^{-1} x = \cos^{-1} a$, where $|a| \le 1$

(viii) $\lim\limits_{x\to a}\tan^{-1} x = \tan^{-1} a$

(ix) $\lim\limits_{x\to\infty}\dfrac{\sin x}{x} = \lim\limits_{x\to\infty}\dfrac{\cos x}{x} = 0$

(x) $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x^m - a^m} = \dfrac{n}{m}a^{n-m}$ and also $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x - a} = na^{n-1}$

Some important Trigonometrical Expansions

(xi) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots\ldots$

(xii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots$

(xiii) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 \ldots\ldots$

(xiv) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots$

(xv) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\ldots$

(xvi) $\tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} - \dfrac{17}{315}x^7 \ldots\ldots$

(xvii) $\sin^{-1} x = x + \dfrac{x^3}{3!} + \dfrac{9x^5}{5!} + \ldots\ldots$

(xviii) $\cos^{-1} x = \dfrac{\pi}{2} - \left(x + \dfrac{x^3}{3!} + \dfrac{9x^5}{5!} + \ldots\ldots\right)$

(xix) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

3. **Logrithmic limits :** To evaluate the logarithmic limits we use following formulae.

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ......$ to ∞ where $-1 < x \le 1$

(ii) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...$ to ∞ where $-1 \le x < 1$

(iii) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(iv) $\lim_{x\to e} \log_e x = 1$

(v) $\lim_{x\to 0} \frac{\log(1-x)}{x} = -1$

(vi) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e,\ a > 0, \ne 1$

4. **Exponential limits**

(i) **Based on series expansion**

We use (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

(b) $a^x = e^{x\log a} = 1 + (x\log a) + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + ...$

To evaluate the exponential limits we use the following results

(a) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(b) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$

(ii) **Based on the form 1^∞ :**

To evaluate the exponential form 1^∞ we use the following results.

(a) $\lim_{x\to 0} (1+x)^{1/x} = e$

(b) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

(c) $\lim_{n\to\infty} x^n = \begin{cases} \infty & if\ \ x > 1 \\ 1 & if\ \ x = 1 \\ 0 & if\ \ -1 < x < 1 \\ \text{does not exist} & if\ \ x \le -1 \end{cases}$ where $n \in N$

(d) $\lim_{n\to\infty} x^{2n} = \begin{cases} \infty & if & x > 1 \\ 1 & if & x = 1 \\ 0 & if & -1 < x < 1 \\ 1 & if & x = -1 \\ \infty & if & x < -1 \end{cases}$

(e) $\lim_{n\to\infty} x^{2n+1} = \begin{cases} \infty & if & x > 1 \\ 1 & if & x = 1 \\ 0 & if & -1 < x < 1 \\ -1 & if & x = -1 \\ -\infty & if & x < -1 \end{cases}$

(f) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$.

Then $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} \left[1 + f(x) - 1\right]^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}$

(iii) **Based on the form 0^0 or ∞^0:**

If $\lim_{x\to a} (f(x))^{g(x)}$ is in the form of 0^0 or ∞^0 then $\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x).\log f(x)}$

5. **L' HOSPITAL'S RULE**

Let $f(x)$ and $g(x)$ be two function differentiable in some neighbourhood of the point *a*, except may be at the point '*a*' itself. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$.

or $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty$. Then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ provided that the limit on the right exist or is $\pm\infty$.

6. **Newton-Leibnitz's formula in evaluating the limits :**

Consider the definite integral, $l(x) = \int_a^{g(x)} f(t)\, dt$, where '*a*' is a given real number.

Newton Leibnitz's formula say that, $\dfrac{dl}{dx} = f(g(x)).\dfrac{d\ g(x)}{dx}$

and if $l(x) = \int_{g(x)}^{h(x)} f(t)\, dt$

$$\frac{dl}{dx} = f(h(x)).\frac{dh(x)}{dx} - f(g(x)).\frac{dg(x)}{dx}$$

Continuity

Chapter 18

CONTINUITY OF A FUNCTION AT A POINT

A function ' f ' is said to be continuous at a point a in the domain of f if the following conditions are satisfied :

(i) $f(a)$ exists (ii) $\lim_{x \to a} f(x)$ exists finitely (iii) $\lim_{x \to a} f(x) = f(a)$.

For the existence of $\lim_{x \to a} f(x)$ it is necessary that $\lim_{x \to a-0} f(x)$ and $\lim_{x \to a+0} f(x)$ both exist finitely and both are equal.

If any one or more of the above condition fail to be satisfied, the function *f* is said to be discontinuous at the point *a*. Geometrically speaking, the graph of the function will exhibit a break at the point $x = a$

GEOMETRICAL MEANING OF CONTINUITY

(i) The function '*f* ' will be continuous at $x = a$ if there is no break in the graph of the function $y = f(x)$ at the point $(a,\ f(a))$.

(ii) The function $f(x)$ will be continuous in the closed interval $[a, b]$ if the graph of $y = f(x)$ is an unbroken line (curved or straight) from the point $(a,\ f(a))$ to $(b,\ f(b))$.

CONTINUITY OF A FUNCTION IN AN OPEN INTERVAL (*a, b*)

A function *f* is said to be continuous in (a, b) if *f* is continuous at each and every point $\in (a, b)$.

CONTINUITY OF A FUNCTION IN A CLOSED INTERVAL [*a, b*]

A function *f* is said to be continuous in a closed interval $[a, b]$ if

(i) f is continuous in the open interval (a, b), and

(ii) f is continuous at '*a*' from the right

i.e., $f(a)$ exists, $\lim_{x \to a+0} f(x)$ exists finitely and $\lim_{x \to a+0} f(x) = f(a)$

(iii) f is continuous at '*b*' from the left

i.e., $f(b)$ exists, $\lim_{x \to b-0} f(x)$ exist finitely and $\lim_{x \to b-0} f(x) = f(b)$.

ALGEBRA OF CONTINUOUS FUNCTIONS

Say $f(x),\ g(x)$ be any two functions having $x = a$ as one of condensation* point of domain.

Let us define $h_1(x) = c_1 f(x) \pm c_2 g(x),\ h_2(x) = f(x)\ .\ g(x),\ h_3(x) = f(x)/g(x)$.

The following results hold true.

Case I : When *f* and *g* both are continuous at $x = a$.

In this case $h_1(x),\ h_2(x)$ and $h_3(x)$ will be continuous at $x = a$, (it has been assumed that $x = a$ is also the condensation point of domain of $h_1(x),\ h_2(x)$ and $h_3(x)$).

Case II : When one of functions either *g* or *f* is discontinuous at $x = a$.

In this case $h_1(x)$ is definitely discontinuous at $x = a$. But nothing can be said, in general about the continuity of $h_2(x)$ and $h_3(x)$ at $x = a$. They may or may not be continuous at $x = a$.

Case III : When *f* and *g* both are discontinuous at $x = a$.

In this case nothing be said, in general, about the continuity of $h_1(x),\ h_2(x)$ and $h_3(x)$ at $x = a$. They may or may not be continuous at $x = a$.

For example, $[x]$ and $\{x\}$ both are discontinuous at all integral points, but their linear combination i.e., $[x]+\{x\}$ is continuous for all values of *x*.

Note : $\max(a, b)=\dfrac{a+b}{2}+\left|\dfrac{a-b}{2}\right|$, $\max\{a, b\}=\dfrac{a+b}{2}-\left|\dfrac{a-b}{2}\right|$.

***ACCUMULATION POINT (Cluster Point or Limit Point or Condensation Point)**

Accumulation point : An accumulation point of a **set of points** is a point P such that there is at least one point of the set distinct from P in any neighborhood of the given point; a point which is the limit of a sequence of points of the set (for spaces ability). An **accumulation point of a sequence** is a point P such that there are an infinite number of terms of the sequence in any neighborhood of P; e.g., the sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, ...$ has two accumulation points the numbers 0 and 1

Note :

(a) Every constant function is everywhere continuous.

(b) The greatest integer function $[x]$ is continuous at all points except at integeral points.

(c) The identity function $I(x)=x, \forall x \in R$ is everywhere continuous.

(d) Modulus function $|x|$ is everywhere continuous.

(e) The exponential function $f(x)=\log_a x, (a>0, a\neq 1)$ is continuous on $(0, \infty)$.

(f) A polynomial function $a_0+a_1x+a_2x^2+...+a_nx^n$, $a_n \neq 0$ and $a_i \in R$, is everywhere continuous.

(g) All trigonometrical functions, namely $\sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x$ are continuous at each point of their respective domains.

(h) Likewise, each inverse trigonometrical function (i.e. $\sin^{-1} x, \cos^{-1} x,$ etc.) is continuous in its domain.

CLASSIFICATION OF DISCONTINUITIES

Although discontinuity means failure of continuity it is interesting to note that all discontinuities are not identical in character. Broadly speaking, discontinuities may be classified as

(i) **Removable,** where at a point $x=a$, $\lim\limits_{x\to a} f(x)$ exists but $f(a)$ is either undefined or if defined, it does not equal to $\lim\limits_{x\to a} f(x)$.

(ii) **Irremovable** where $\lim\limits_{x\to a} f(x)$ does not exist, even though $f(a)$ exists or additionally $f(a)$ also does not exist.

In the first case the removal of discontinuity is achieved by modifying the definition of the function suitably so that $f(a)$ is equal to $\lim\limits_{x\to a} f(x)$ which is found existing.

In the second case when $\lim\limits_{x\to a} f(x)$ does not exist, no modification of definition at $x=a$ can succeed to make the $\lim\limits_{x\to a} f(x)$ exist. Hence the discontinuity remains irremovable.

Removable type of discontinuity can be further classified as :

(i) **Missing point discontinuity** (ii) **Isolated point discontinuity.**

(i) In the case of function '*f*' where $\lim\limits_{x\to a} f(x)$ exists finitely but $f(a)$ is not defined, the graph of the function will show an imperceptible break at $x=a$ in as much as the point $x=a$ is missing on it. It just like a wire (of the shape of the graph) which is broken but whose ends are put together.

Appropriately, such a discontinuity is called missing point discontinuity This type of discontinuity can, however, be removed by redefining the function such that $f(a)=\lim_{x\to a} f(x)$; thus the redefinition provides welding together the broken wire at the point of break.

(ii) In case a function f is defined such that $\lim_{x\to a} f(x)\neq f(a)$, the graph of the function will show a break at $x=a$ together with a single point at $x=a$ isolated from the main graph, thus justifying the name isolated point discontinuity.

Irremovable type of discontinuity can be further classified as :

(i) **Finite discontinuity**
(ii) **Infinite discontinuity**
(iii) **Oscillatory discontinuity.**

In all these cases the value $f(a)$ of the function at $x=a$ (point of discontinuity) may exist or may not exist but $\lim_{x\to a} f(x)$ does not exist.

(i) **Finite discontinuity:** Here $f(a)$ exists finitely, but it is either equal to $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ are finite and unequal.

(ii) **Infinite discontinuity:** A discontinuity is termed infinite, if the magnitude of the break at a point of discontinuity is infinite or the break occurs at infinity. Here both $\lim_{x\to a} f(x)$ and $f(a)$ do not exist.

e.g. for $f(x)=\dfrac{1}{x-4}$, $f(4)$ is undefined. $\dfrac{1}{x-4}\to\infty$ for $x\to 4^-$ and $\dfrac{1}{x-4}\to\infty$ for $x\to 4^+$

Showing $x=4$, is a point of discontinuity occurring at ∞, since both branches of the graph from the left as well as from the right of the line $x=4$ tend towards ∞,

(iii) **Oscillatory discontinuity:** For a function *f*, if $f(a)$ is undefined as $f(x)$ oscillates finitely as $x\to a$ then $x=a$ is called a point of finite oscillatory discontinuity, and incase $f(x)$ oscillates infinitely as $x\to a$, then $x=a$ is called a point of infinite oscillatory discontinuity.

PROPERTIES OF CONTINUOUS FUNCTIONS

Here we present two extremely useful properties of continuous functions ;

Let $y=f(x)$ be a continuous function $\forall\ x\in[a,b]$, then following results hold true.

(i) *f* is bounded between *a* and *b*. This simply means that we can find real numbers m_1 and m_2 such $m_1\le f(x)\le m_2 \quad \forall\ x\in[a,b]$.

(ii) Every value between $f(a)$ and $f(b)$ will be assumed by the function atleast once. This property is called intermediate value theorem of continuous function.

In particular if $f(a).\ f(b)<0$, then $f(x)$ will become zero atleast once in (a,b). It also means that if $f(a)$ and $f(b)$ have opposite signs then the equation $f(x)=0$ will have atleast one real root in (a,b).

Differentiation Chapter 19

DEFINITION

If $f(x)$ is a function and a and $a+h$ belongs to the domains of f, then the limit given by $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$, if it finitely exist, is called the derivative of $f(x)$ with respect to (or w.r.t) x at $x=a$ and is denoted by $f'(a)$,

$$\therefore\ f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Note : $f'(a)$ is the derivates of $f(x)$ w.r.t x at $x=a$.

DERIVATIVE OF *f* (*x*) FROM THE FIRST PRINCIPLES (*i.e.* definition or ab-initio)

Let $y=f(x)$ …(1)

be a given function defined in some domain.

Let δx be small change in x, and δy be the corresponding change in *y*.

$\therefore\ y+\delta y=f(x+\delta x)$ …(2)

On subtracting (1) from (2), we have

$\therefore\ \delta y=f(x+\mathrm{d}x)-f(x)$

Dividing by $\delta x\neq 0$, $\dfrac{\delta y}{\delta x}=\dfrac{f(x+\delta x)-f(x)}{\delta x}$

Taking limits on both side as $\delta x\to 0$, we get

$$\lim_{\delta\to 0}\frac{\delta y}{\delta x} = \lim_{\delta\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$$

if it finitely exist (i.e. if f is derivable at *x*) is called the differential coefficient (d.c.) or the derivative of $f(x)$ w.r.t. x or derived function

Denoting L.H.S by $\dfrac{dy}{dx}=\lim\limits_{\delta x\to 0}\dfrac{\delta y}{\delta x}$, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x}=f'(x)$ and it may be denoted by anyone of the following symbol :

$$f'(x),\ \frac{dy}{dx},\ \frac{d}{dx}(y),\ \frac{d}{dx}\big(f(x)\big),\ y',\ y_1,\ D_x(y).$$

The **general derivative** of f w.r.t. *x* is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The denominator '*h*' represents the change (increment) in the value of the *x* whenever it changes from *x* to $x+h$. The numerator represents the corresponding change (increment) in the value of $f(x)$. Hence we can write

$$f'(x) = \lim_{h\to 0}\frac{\text{change in } f(x) \text{ or } y}{\text{increment in } x}.$$

DIFFERENTIABILITY

(i) A function f is said to have left hand derivative at $x = a$ iff f is defined in some (undeleted) left neighbourhood of a and $\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$ exists finitely and its value is called the left hand derivative at a and is denoted by $f'(a^-)$.

(ii) A function f is said to have right-hand derivative at $x = a$ iff f is defined in some (undeleted) right neighbourhood of a and $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ exists finitely and its value is called the right hand derivative at a and is denoted by $f'(a^+)$.

(iii) A function f is said to have a derivative (or is differentiable) at a if f is defined in some (undeleted) neighbourhood of a and $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists finitely and its value is called the derivative or differential coefficient of f at a and is denoted by $f'(a)$ or $\left.\frac{df(x)}{dx}\right|_{x=a}$

(iv) If a function f is differentiable at a point 'a' then it is also continuous at the point 'a'. But, converse may not be true. For example, $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable at $x = 0$.

(v) A function f is differentiable at a point $x = a$ and $P(a, f(a))$ is the corresponding point on the graph of $y = f(x)$ iff the curve does not have P as a corner point.

Note : From (iv) and (v) it is clear that if a function f is not differentiable at a point $x = a$ then either the function f is not continuous at $x = a$ or the curve represented by $y = f(x)$ has a corner at the point $(a, f(a))$ (i.e. the curve suddenly changes the direction)

(vi) A function f is differentiable (or derivable) on $[a, b]$ if

(a) f is continuous at every point of (a, b)

(b) $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ and $\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$ both exist.

A function f is said to be differentiable if it is differentiable at every point of the domain.

A function f is said to be everywhere differentiable if it is differentiable for each $x \in R$.

SOME STANDARD RESULTS ON DIFFERENTIABLITY

(i) Every polynomial function, every exponential function $a^x \ (a > 0)$ and every constant function are differentiable at each $x \in R$.

(ii) The logarithmic functions, trigonometrical functions and inverse – trigonometrical functions are always differentiable in their domains.

(iii) The sum, difference, product and quotient (under condition) of two differentiable functions is differentiable.

(iv) The composition of differentiable functions (under condition) is a differentiable function.

DERIVATIVES OF SOME STANDARD FUNCTIONS

(i) $\frac{d}{dx}(c) = 0$ if c is a constant and conversely also.

Test of constancy. If at all points of a certain interval $f'(x)=0$, then the function f is constant in that interval.

(ii) $\frac{d}{dx}(x^n)=nx^{n-1}$

(iii) $\frac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}.a$

(iv) $\frac{d}{dx}(\sin x)=\cos x$

(v) $\frac{d}{dx}(\cos x)=-\sin x$

(vi) $\frac{d}{dx}(\tan x)=\sec^2 x$

(vii) $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$

(viii) $\frac{d}{dx}(\sec x)=\sec x\tan x$

(ix) $\frac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x\cot x$

(x) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$

(xi) $\frac{d}{dx}(\cos^{-1} x)=\frac{-1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$

(xiii) $\frac{d}{dx}(\cot^{-1} x)=\frac{-1}{1+x^2}$

(xiv) $\frac{d}{dx}(\sec^{-1} x)=\frac{1}{|x|\sqrt{x^2-1}}$

(xv) $\frac{d}{dx}(\text{cosec}^{-1} x)=\frac{-1}{|x|\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}(a^x)=a^x\log_e a$

(xvii) $\frac{d}{dx}(e^x)=e^x$

(xviii) $\frac{d}{dx}(\log_a x)=\frac{1}{x\log_e a}$

$\frac{d}{dx}(\log_e x)=\frac{1}{x}$

(xix) $\frac{d}{dx}(|x|)=\frac{x}{|x|},\ x\neq 0,\quad y=|x|$ is not differentiable at $x=0$

(xx) $\frac{d}{dx}([x])=\begin{cases}0 \text{ for } x\in R-I\\ \text{does not exist for } x\in I\end{cases}$

(xxi) $\frac{d}{dx}(\{x\})=\begin{cases}1 \text{ if } x\in R-I\\ \text{does not exist if } x\in I\end{cases}$

SOME RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero , i.e. $\frac{d}{dx}(c)=0.$

2. The derivative of constant times a function is constant times the derivative of the function, i.e.

$$\frac{d}{dx}\{c.f(x)\}=c\cdot\frac{d}{dx}\{f(x)\}.$$

3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

$$\frac{d}{dx}\{f(x)\pm g(x)\}=\frac{d}{dx}\{f(x)\}\pm\frac{d}{dx}\{g(x)\}.$$

4. **PRODUCT RULE OF DIFFERENTIATION**

The derivative of the product of two functions $=$ (first function) $\times$ (derivative of second function) $+$ (second function) $\times$ (derivative of first function)

i.e. $\frac{d}{dx}\{f(x).g(x)\}=f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}$

5. **QUOTIENT RULE OF DIFFRENTIATION**

The derivative of the quotient of two functions

$$= \frac{(\text{denom.} \times \text{ derivative of num.}) - (\text{num.} \times \text{ derivative of denom.})}{(\text{denominator})^2}$$

i.e. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$

6. **DERIVATIVE OF A FUNCTION OF A FUNCTION (CHAIN RULE)**

If y is a differentiable function of t and t is a differentiable function of x i.e. $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$.

Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$, then, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$.

7. **DERIVATIVE OF PARAMETRIC FUNCTIONS**

Sometimes x and y are separately given as functions of a single variable t (called a parameter) i.e. $x = f(t)$ and $y = g(t)$.

In this case,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}. \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}.$$

8. **DIFFERENTIATION OF IMPLICIT FUNCTIONS**

If in an equation, x and y both occurs together i.e. $f(x, y) = 0$ and this equation can not be solved either for y or x, then y (or x) is called the implicit function of x (or y).

For example $x^3 + y^3 + 3axy + c = 0$, $x^y + y^x = a$ etc.

Working rule for finding the derivative

First Method :

(i) Differentiate every term of $f(x, y) = 0$ with respect to x.

(ii) Collect the coefficients of $\frac{dy}{dx}$ and obtain the value of $\frac{dy}{dx}$.

Second Method :

If $f(x, y) = \text{constant}$, then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial differential coefficients of $f(x, y)$ with respect to x and y respectively.

9. **DIFFERENTIATION OF LOGARITHMIC FUNCTIONS**

When base and power both are the functions of x *i.e.*, the function is of the form $[f(x)]^{g(x)}$.

$$y = [f(x)]^{g(x)}$$

$$\log y = g(x)\log[f(x)]$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}g(x).\log[f(x)]$$

$$\frac{dy}{dx} = [f(x)]^{g(x)}.\left\{\frac{d}{dx}[g(x)\log f(x)]\right\}.$$

10. DIFFERENTIATION BY TRIGONOMETRICAL SUBSTITUTIONS

Some times before differentiation, we reduce the given function in a simple form using suitable trigonometrical or algebraic transformations.

	Function	**Substitution**	
(i)	$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$	where $\theta\in[0,\pi/2]$
(ii)	$\sqrt{x^2+a^2}$	$x=a\tan\theta$ or $a\cot\theta$	where $\theta\in(0,\pi/2)$
(iii)	$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $a\operatorname{cosec}\theta$	where $\theta\in(0,\pi/2)$
(iv)	$\sqrt{\dfrac{a-x}{a+x}}$	$x=a\cos 2\theta$	where $\theta\in[0,\pi/2]$
(v)	$\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$	$x^2=a^2\cos 2\theta$	where $\theta\in[0,\pi/2]$
(vi)	$\sqrt{ax-x^2}$	$x=a\sin^2\theta$	where $\theta\in[0,\pi/2]$
(vii)	$\sqrt{\dfrac{x}{a+x}}$	$x=a\tan^2\theta$	where $\theta\in(0,\pi/2)$
(viii)	$\sqrt{\dfrac{x}{a-x}}$	$x=a\sin^2\theta$	where $\theta\in[0,\pi/2]$
(ix)	$\sqrt{(x-a)(x-b)}$	$x=a\sec^2\theta-b\tan^2\theta$	where $\theta\in(0,\pi/2)$
(x)	$\sqrt{(x-a)(b-x)}$	$x=a\cos^2\theta+b\sin^2\theta$	where $\theta\in[0,\pi/2]$

11. DIFFERENTIATION OF INFINITE SERIES

(i) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.....\infty}}}$ then

$$\Rightarrow \quad y=\sqrt{f(x)+y} \quad \Rightarrow \quad y^2=f(x)+y$$

$$2y\frac{dy}{dx}=f'(x)+dy/dx$$

$$\therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$$

(ii) If $y=f(x)^{f(x)^{f(x)\ldots\infty}}$ then $y=f(x)^y$.

$$\therefore \log y=y\log\left[f(x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{y\cdot f'(x)}{f(x)}+\log f(x)\cdot\left(\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx}=\frac{y^2 f'(x)}{f(x)\left[1-y\log f(x)\right]}$$

(iii) If $y=f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x).....}}}$

Then $\dfrac{dy}{dx} = \dfrac{y\ f'(x)}{2y - f(x)}$.

12. $f''(\alpha) = \lim\limits_{h\to 0} \dfrac{f(a+2h) - 2f(a+h) + f(a)}{h^2}$ and in general

$$f^{(n)}(\alpha) = \lim_{h\to 0} \frac{f(\alpha+nh) - {}^nC_1 f(\alpha+(n-1)h) + {}^nC_2 f(\alpha+(n-2)h) + ... + (-1)^n f(\alpha)}{h^n}$$

13. $\left\{\dfrac{d}{dx} f^{-1}(x)\right\}_{x=f(\alpha)} = \dfrac{1}{\left\{\dfrac{d}{dx} f(x)\right\}_{x=\alpha}}$

ALGEBRA OF DIFFERENTIABLE FUNCTIONS

(i) **Logarithmic differentiation**

If $y = f_1(x)\ f_2(x)$ or $y = f_1(x)\, f_2(x)\, f_3(x)....$

or $y = \dfrac{f_1(x)\, f_2(x).....}{g_1(x)\, g_2(x).....}$, then first take log on both sides and then differentiate.

If u, v are functions of x, then $\dfrac{d}{dx}(u^v) = u^v \dfrac{d}{dx}(v \log u)$

In particular, $\dfrac{d}{dx}(x^x) = x^x(1+\log x)$

(ii) $\dfrac{d}{dx}(|u|) = \dfrac{u}{|u|}\dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(\log f(x)) = \dfrac{1}{f(x)}\dfrac{d}{dx} f(x)$

(iv) $\dfrac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \log a\,.\, f'(x)$.

LEIBNITZ THEOREM AND *n*TH DERIVATIVES

Let $f(x)$ and $g(x)$ be functions both possessing derivatives up to nth order. Then,

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = f^n(x)g(x) + {}^nC_1 f^{n-1}(x)g^1(x) + {}^nC_2 f^{n-2}(x)g^2(x) + ... +$$
$${}^nC_r f^{n-r}(x)g^r(x) + ... + {}^nC_n f(x)g^n(x).$$

$$\frac{d^n}{dx^n}(x^n) = n!;\ \frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}};\ \frac{d^n}{dx^n}(\sin x) = \sin\left(x + n\frac{\pi}{2}\right),$$

$$\frac{d^n}{dx^n}(\cos x) = \cos\left(x + n\frac{\pi}{2}\right);\ \frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}.$$

SUCCESSIVE DIFFERENTIATION

(i) If $y = (ax+b)^m$, $m \notin N$, then $y_n = m(m-1)(m-2).....(m-n+1)(ax+b)^{m-n}.a^n$

(ii) If $y = (ax+b)^m$, $m \in N$, then

$$y_n = \begin{cases} m(m-1)(m-2).....(m-n+1)(ax+b)^{m-n}.a^n \text{ for } n < m \\ m!a^m, \text{ if } n = m \\ 0, \qquad n > m \end{cases}$$

(iii) If $y=\frac{1}{ax+b}$, then $y_n=(-1)^n.n!(ax+b)^{-n-1}.a^n$

(iv) If $y=\log(ax+b)$, then $y_n=(-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$

(v) If $y=\sin(ax+b)$, then $y_n=a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$

(vi) If $y=\cos(ax+b)$, then $y_n=a^n\cos\left(ax+b+n\frac{\pi}{2}\right)$

(vii) If $y=a^x$ then $y_n=a^x(\log_e a)^n$.

PARTIAL DIFFERENTIATION

The partial differential coefficient of $f(x, y)$ with respect to *x* is the ordinary differential coefficient of $f(x, y)$ when *y* is regarded as a constant. It is written as $\frac{\partial f}{\partial x}$ or $D_x\ f$ or f_x.

Thus, $\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h, y)-f(x, y)}{h}$

Again, the partial differential coefficient $\frac{\partial f}{\partial y}$ of $f(x, y)$ with respect to *y* is the ordinary differential coefficient of $f(x, y)$ when *x* is regarded as a constant.

Thus, $\frac{\partial f}{\partial y}=\lim_{k\to 0}\frac{f(x, y+k)-f(x, y)}{k}$

e.g., If $z=f(x, y)=x^4+y^4+3xy^2+x^2y+x+2y$

Then $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x=4x^3+3y^2+2xy+1$ (Here *y* is regarded as constant)

$\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_x=4y^3+6xy+x^2+2$ (Here *x* is regarded as constant)

HIGHER PARTIAL DERIVATIVES

Let $f(x, y)$ be a function of two variables such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ both exist.

(i) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. '*x*' is denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

(ii) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. '*y*' is denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}.

(iii) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. '*y*' is denoted by $\frac{\partial^2 f}{\partial x\,\partial y}$ or f_{xy}.

(iv) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. '*x*' is denoted by $\frac{\partial^2 f}{\partial y\,\partial x}$ or f_{yx}.

These four are second order partial derivatives.

Note : If $f(x, y)$ possesses continuous partial derivatives then in all ordinary cases.

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{or} \quad f_{xy} = f_{yx}.$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If $f(x, y)$ is a homogeneous function in x, y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

DEDUCTION OF EULER'S THEOREM

If $f(x, y)$ is a homogeneous function in x, y of degree n, then

(i) $x\dfrac{\partial^2 f}{\partial x^2} + y\dfrac{\partial^2 f}{\partial x\,\partial y} = (n-1)\dfrac{\partial f}{\partial x}$

(ii) $x\dfrac{\partial^2 f}{\partial y\,\partial x} + y\dfrac{\partial^2 f}{\partial y^2} = (n-1)\dfrac{\partial f}{\partial y}$

(iii) $x\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x\,\partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)\,f(x, y)$

Application of Derivatives **Chapter 20**

I. APPLICATION IN MECHANICS & RATE MEASURER

VELOCITY AND ACCELERATION IN RECTILINEAR MOTION

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of its velocity with respect to time.
Let velocity and acceleration at time *t* be *v* and *a* respectively.

Then, Velocity $(v) = \dfrac{ds}{dt}$;

Acceleration $(a) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$.

DERIVATIVE AS THE RATE OF CHANGE

If a variable quantity *y* is some function of time *t i.e.*, $y = f(t)$, then for a small change in time Δt we have a corresponding change Δy in *y*.

Thus, the average rate of change $= \dfrac{\Delta y}{\Delta t}$.

The differential coefficient of *y* with respect to *x*, *i.e.*, $\dfrac{dy}{dx}$ is nothing but the rate of change of *y* relative to *x*.

RATE OF CHANGE OF QUANTITY

1. **Chain Rule.** If both x and y are functions of the parameter t, then $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$
2. If the rate of change of a variable is positive (negative) then the value of the variable increases (decreases) with the increase in the value of independent variable.

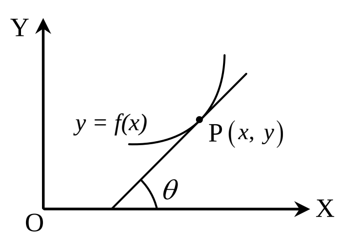
ERROR AND APPROXIMATION

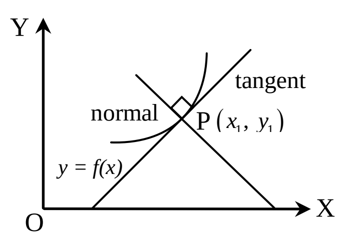
1. **Approximate value** of a function $f(x+h) = f(x) + hf'(x)$
2. **Absolute error :** The error Δx in x is called, the absolute error.
3. **Relative error :** If Δx is error in x, then the ratio $\dfrac{\Delta x}{x}$ is called relative error.
4. **Percentage error :** If $\dfrac{\Delta x}{x}$ is relative error, then $\dfrac{\Delta x}{x} \times 100$ is called percentage error in x.

II. TANGENTS AND NORMALS

1. Geometrical interpretation of $\dfrac{dy}{dx}$

   If $P(x_1, y_1)$ is a point on the curve $y = f(x)$, then value of $\dfrac{dy}{dx}$ at P gives the slope of tangent to the curve at P.





2. Equation of tangent to the curve at P is $y - y_1 = \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

3. If $\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}$ is zero, then the tangent to the curve $y = f(x)$ at P is $y = y_1$ which is parallel to *x*-axis.

   The equation of normal to the curve at P is given by $x = x_1$ which is parallel to *y*-axis

4. If $\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)} \neq 0,$ then slope of normal at P is $-\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}}$ and equation of normal is

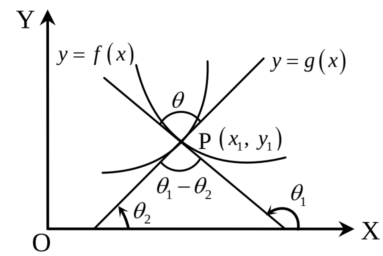
$$y - y_1 = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}}(x - x_1)$$

5. If $\frac{dx}{dy} = 0,$ then the tangent is perpendicular to *x*-axis and its equation is $x = x_1$ or normal is parallel to *x*-axis and equation of normal is $y = y_1$

ANGLE OF INTERSECTION OF TWO CURVES

Angle of intersection of two curves is the (acute) angle between the tangents to the two curves at their point of intersection.

If θ is the acute angle between the tangents, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$



where $m_1 =$ value of $\frac{dy}{dx}$ at the common point for first curve.

and $m_2 =$ value of $\frac{dy}{dx}$ at the common point for the second curve.

If θ is the required angle of intersection, then, $\theta = |(\theta_1 - \theta_2)|,$

where θ_1 and θ_2 are the inclination of tangent to the curves $y = f(x)$ and $y = g(x)$ respectively at the point P.

ORTHOGONAL AND TOUCHING CURVES

Two curves are said to be orthogonal (or intersect orthogonally) if the angle of intersection of two curves is a right angle. i.e. if $m_1 m_2 = -1$

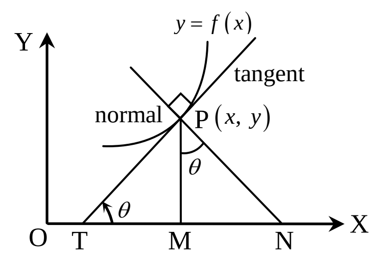
Two curves touch each other if $m_1 = m_2$

Note : The curve $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ cut each other orthogonally if $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

LENGTH OF TANGENT, NORMAL, SUB-TANGENT AND SUBNORMAL

Let the tangent and normal at the point $P(x,\, y)$ on the curve meet the axis of x at the points T and N respectively. Let M be the foot of the ordinates at P. Then,

(i) Length of the tangent $= PT = |y \operatorname{cosec}\theta|$



$$= \left|y\sqrt{1 + \cot^2\theta}\right| = \left|\frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right| = \left|y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\right|$$

(ii) Length of the normal $= PN = \left| y \sec\theta \right| = \left| y\sqrt{1+\tan^2\theta} \right| = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$

(iii) Length of subtangent $= TM = \left| y \cot\theta \right| = \left| \frac{y}{\left(\frac{dy}{dx}\right)} \right| = \left| y\, \frac{dx}{dy} \right|$

(iv) Length of subnormal $= MN = \left| y \tan\theta \right| = \left| y\left(\frac{dy}{dx}\right) \right|$.

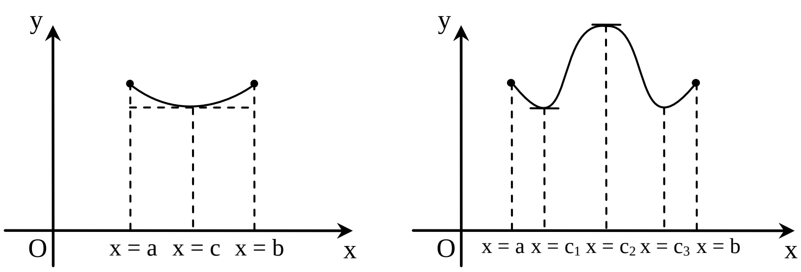
III. ROLLE'S AND LAGRANGE'S MEAN VALUE THEOREMS

ROLLE'S THEOREM

If a function $f(x)$ defined on $[a, b]$ is

(i) continuous on $[a, b]$, (ii) differentiable on (a, b) and

(iii) $f(a) = f(b)$, then there exists at least one point c, $a < c < b$ such that $f'(c) = 0$.



(i) (ii)

There is one point c in figure (i) and more than one point c in figure (ii).

Geometrical Interpretation : If a function $f(x)$ satisfies all the above three conditions, then there exists at least one point c between a and b at which tangent to the curve is parallel to x-axis.

Algebraic Interpretation : If $f(x)$ is a polynomial with a, b roots of $f(x) = 0$, i.e. $f(a) = 0 = f(b)$ then $f(x)$ satisfies all the three conditions of Rolle's theorem. Therefore, there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$ i.e. c is a root of $f'(x) = 0$

Thus, we have **Rolle's theorem for polynomials.** If $f(x)$ is a polynomial, then between any two roots of $f(x) = 0$, there always lies a root of $f'(x) = 0$

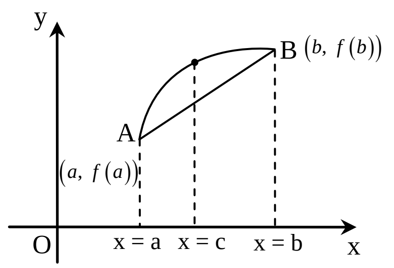
LAGRANGE'S MEAN VALUE THEOREM

If a function $f(x)$ defined on $[a, b]$ is

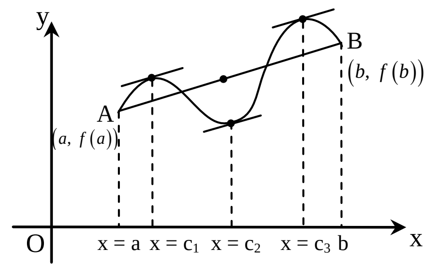
(i) continuous on $[a, b]$,

(ii) differentiable on (a, b), then there exists at least one point $c, a < c < b$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Geometrical Interpretation : If a function $f(x)$ satisfies the above two conditions, then there exists at least one point c between a and b at which tangent is parallel to the chord joining the point $A\left(a,\ f(a)\right)$ and $B\left(b,\ f(b)\right)$.



(i)



(ii)

Note :

(i) A polynomial functions is everywhere continuous as well as differentiable.

(ii) An exponential function, sine and cosine functions are everywhere continuous as well as differentiable.

(iii) Logarithmic function is continuous as well as differentiable in its domain.

(iv) $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$ are discontinuous at those points where $\cos x = 0$ i.e. $x = (2n+1)\dfrac{\pi}{2}$.

(v) $|x|$ is not differentiable at $x = 0$.

(vi) If $f'(x) \to \pm\infty$ as $x \to a$, then $f(x)$ is not differentiable at $x = a$.

For example, if $f(x) = (2x-1)^{1/2}$,

Then $f'(x) = \dfrac{1}{\sqrt{2x-1}} \to \infty$ as $x \to \left(\dfrac{1}{2}\right)^{+}$. So, $f(x)$ is not differentiable at $x = \dfrac{1}{2}$

and $\cot x = \dfrac{\cos x}{\sin x}$ and $\operatorname{cosec} x = \dfrac{1}{\sin x}$

are discontinuous at those points where $\sin x = 0$ i.e. $x = n\pi$.

(vii) The sum, difference, product and quotient of continuous (differentiable) functions is continuous (differentiable) (with Denominator $\neq 0$ in last case).

IV. INCREASING AND DECREASING FUNCTIONS (MONOTONICITY)

1. A function $f(x)$ is said to be increasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1,\ x_2 \in I$. A function $f(x)$ is said to be decreasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, for all $x_1,\ x_2 \in I$.

A function $f(x)$ is said to be strictly increasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, for all $x_1,\ x_2 \in I$.

A function $f(x)$ is said to be strictly decreasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1,\ x_2 \in I$.

A function $f(x)$ is said to be monotonic on an interval I, if it is either increasing or decreasing on I.

A function $f(x)$ is increasing (decreasing) at a point x_0, if $f(x)$ is increasing (decreasing) on an interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. A function $f(x)$ is increasing (decreasing) on $[a, b]$, if it is increasing (decreasing) on (a, b) and it is also increasing (decreasing) at $x = a$ and $x = b$

2. **TEST OF MONOTONICITY**

Let $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f(x)$ is increasing $(\uparrow)$ on $[a, b]$ then $f'(x) \geq 0$ for all $x \in (a, b)$

(b) If $f(x)$ is decreasing $(\downarrow)$ on $[a, b]$ then $f'(x) \leq 0$ for all $x \in (a, b)$

(c) If $f'(x) > 0$ for all $x \in (a, b)$ then $\uparrow^s$

(d) If $f'(x) < 0$ for all $x \in (a, b)$ then $\downarrow_s$

(e) If a function $f(x)$ is defined on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then $f(x)$ is strictly increasing $(\uparrow^S)$ on (a, b)

(f) If a function $f(x)$ is defined on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then $f(x)$ is strictly decreasing $(\downarrow_S)$ on (a, b)

V. MAXIMA AND MINIMA

1. A real valued function 'f' with domain D_f is said to have absolute maximum at a point $x_0 \in D_f$ iff $f(x_0) \geq f(x)$ $\forall x \in D_f$ and $f(x_0)$ is called the absolute maximum value and x_0 is called the point of absolute maxima.

Likewise, 'f' is said to have absolute minimum at a point x_0, if $f(x_0) \leq f(x) \forall x \in D_f$ and x_0 is called the point of absolute minima.

Note : Absolute maximum and Absolute minimum values of a function, if exist are unique and may occur at more than one point of D_f.

2. A real valued function 'f' with domain D_f is said to have local maxima at $x_0 \in D_f$, if for some positive δ, $f(x_0) > f(x)$ $\forall x \in (x_0 - \delta, x_0 + \delta)$. $f(x_0)$ is called the local maximum value and $(x_0, f(x_0))$ is called the point of local maxima.

Likewise 'f' is said to have local minimum at $x_0 \in D_f$ if for some positive δ, $f(x_0) < f(x) \forall x \in (x_0 - \delta, x_0 + \delta)$. $f(x_0)$ is called the local minimum value and $(x_0, f(x_0))$ is called the point of local minima.

3. A point of domain of 'f' is an extreme point of f, if it is either a point of local maxima or local minima. It is also called as turning point.

A point x_0 of domain of 'f' is a critical point, if either f is not differentiable at x_0 or $f'(x_0) = 0$.

A point x_0 where $f'(x_0) = 0$ is called a stationary point and $f(x_0)$ is called the stationary value at x_0.

Note : A local maxima or minima may not be absolute maxima or absolute minima. A local maximum value at some point may be less than local minimum value of the function at another point.

4. **To find absolute maximum and absolute minimum in [a, b], proceed as follows**

(i) Evaluate $f(x)$ at points where $f'(x) = 0$

(ii) Evaluate $f(x)$ at points where $f'(x)$ does not exist

(iii) Find $f(a)$ and $f(b)$

Then, the maximum of these values is the absolute maximum value and minimum of these values is the absolute minimum value of the function f.

But if the given function has domain (a, b) then we will find $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$, but note that $\lim_{x\to a^+} f(x)$ is greatest (or least) then we will say that point of Absolute Maxima (or Absolute Minima) does not exist, similarly for $\lim_{x\to b^-} f(x)$.

5. **To find the local maximum and local minimum.**

First Derivative Test

A point x_0 is a point of local maxima (local minima) if

(i) $f'(x_0) = 0$ and

(ii) $f'(x)$ changes sign from positive (negative) to negative (positive) while passing through x_0

Second Derivative Test

A point x_0 is a point of local maxima (local minima) if

(i) $f'(x_0) = 0$ and

(ii) $f''(x_0) < 0 (> 0)$

If $f''(x_0) = 0$, then second derivative test fails and find $f'''(x_0)$. If $f'''(x_0) \neq 0$ then it is a point of inflexion. If $f'''(x_0) = 0$, find $f^{(iv)}(x_0)$. If $f^{iv}(x_0)$ is < 0 then x_0 is the point of local maxima. If $f^{iv}(x_0) > 0$ then x_0 is the point of local minima. If $f^{iv}(x_0) = 0$ then we find $f^{v}(x_0)$. If $f^{v}(x_0) \neq 0$ then x_0 given the point of inflexion. If $f^{v}(x_0) = 0$ then we find $f^{vi}(x_0)$ is so on in similar way.

6. If $f'(x_0)$ does not exist, but f' exists in a neighbourhood of 'x_0', then

x	slightly $< x_0$	slightly $> x_0$	Nature of the point 'x_0'
$f'(x)$	$+$ve	$-$ve	Maxima
$f'(x)$	$-$ve	$+$ve	Minima

PROPERTIES OF MONOTONIC FUNCTIONS

(i) If $f(x)$ is strictly increasing on $[a, b]$, then f^{-1} exists and is also strictly increasing.

(ii) If $f(x)$ is strictly increasing on $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

(iii) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) on $[a, b]$ then $(gof)(x)$ is also monotonically (or strictly) increasing (or decreasing) on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly (monotonically) increasing and the other is strictly (monotonically) decreasing, then $(gof)(x)$ is strictly (monotonically) decreasing on $[a, b]$.

CONCAVITY. CONVEXITY AND POINT OF INFLEXION

1. **Concavity and Convexity**

Let P be a point on the curve $y = f(x)$, where the tangent PT is not parallel to y-axis. The curve is said to be concave upwards (or convex downwards) at P if in the immediate neighbourhood of P, the curve lies above the tangent PT on both sides (as in figure I).

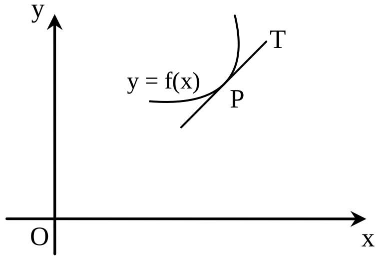


Figure I

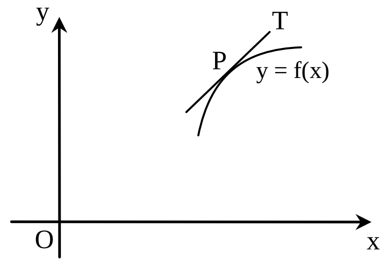


Figure II

The curve is said to be concave downwards (or convex upwards) at P if in the immediate neighbourhood of P, the curve lies below the tangent PT on both sides (as in figure II).

Criteria for concavity and convexity

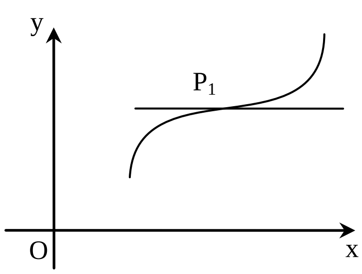
At a point P on the curve $y = f(x)$, the curve is

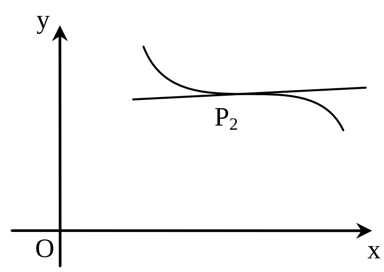
(i) Convex downward if $f''(x) > 0$

(ii) Concave downwards if $f''(x) < 0$.

2. **Point of inflexion**

A point on a curve at which the curve changes from concavity to convexity or vice-versa is called a point of inflexion. At a point of inflexion, the tangent to the curve crosses the curve.





Criteria for a point of inflexion

A point P is a point of inflexion if

(i) $\dfrac{d^2y}{dx^2} = 0$ at this point, and

(ii) $\dfrac{d^2y}{dx^2}$ changes sign in passing through this point.

Indefinite Integral Chapter 21

SOME BASIC INTEGRALS

1. $\int (f(x))^n f'(x)\,dx = \dfrac{(f(x))^{n+1}}{n+1} + C$ *for* $n \neq -1$. In particular, $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ *for* $n \neq -1$.

2. $\int \dfrac{f'(x)}{f(x)}dx = \log|f(x)| + C$. In particular, $\int \dfrac{1}{x}dx = \log|x| + C$.

3. $\int \sin x dx = -\cos x + C$.

4. $\int \cos x dx = \sin x + C$.

5. $\int \tan x dx = -\log|\cos x| + C = \log|\sec x| + C$.

6. $\int \cot x dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$.

7. $\int \sec x dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right| + C$

8. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\dfrac{x}{2}\right| + C$

9. $\int a^x dx = \dfrac{a^x}{\log_e a} + C$. In particular, $\int e^x dx = e^x + C$.

10. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$. In particular, $\int \dfrac{dx}{1 + x^2} = \tan^{-1} x + C$.

11. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$.

12. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$.

13. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\dfrac{x}{a} + C$.

14. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a}$

15. $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$.

16. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$.

17. $\int \sqrt{a^2 - x^2}\,dx = \dfrac{1}{2}x\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C$.

18. $\int \sqrt{a^2 + x^2}\,dx = \dfrac{1}{2}x\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$.

19. $\int \sqrt{x^2 - a^2}\,dx = \dfrac{1}{2}x\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$.

20. $\int |x|^n dx = \dfrac{x|x|^n}{n+1} + C$, where $n \neq -1$

INTEGRALS OF THE FORM $\int \mathbf{e^x \{f(x) + f'(x)\} dx}$:

If the integral is of the form $\int e^x \{f(x) + f'(x)\}dx$, then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get

1. $\int e^x [f(x) + f'(x)]dx = e^x f(x) + c$

2. $\int e^{mx}\left[mf(x)+f'(x)\right]dx = e^{mx}f(x)+c$

3. $\int e^{mx}\left[f(x)+\dfrac{f'(x)}{m}\right]dx = \dfrac{e^{mx}f(x)}{m}+c$

INTEGRAL IS OF THE FORM $\int\left[\mathbf{xf'(x)+f(x)}\right]\mathbf{dx}$:

If the integral is of the form $\int\left[xf'(x)+f(x)\right]dx$ then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is , by doing so, we get,

$$\int\left[x\,f'(x)+f(x)\right]dx = x\,f(x)+c.$$

INTEGRALS OF THE FORM $\int \mathbf{e^{ax}sinbxdx}$, $\int \mathbf{e^{ax}cosbx\,dx}$:

$$\int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^{ax}.\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^{ax}.\sin(bx+c)\ dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c)-b\cos(bx+c)\right]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$$

$$\int e^{ax}.\cos(bx+c)\ dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c)+b\sin(bx+c)\right]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$$

INTEGRALS OF THE FORM $\int\dfrac{\mathbf{x^2+1}}{\mathbf{x^4+kx^2+1}}\mathbf{dx}$, $\int\dfrac{\mathbf{x^2-1}}{\mathbf{x^4+kx^2+1}}\mathbf{dx}$, $\int\dfrac{\mathbf{dx}}{\mathbf{x^4+kx^2+1}}$, where $\mathbf{k\in R}$

Working Method

(a) To evaluate these types of integrals divide the numerator and denominator by x^2

(b) Put $x+\dfrac{1}{x}=t$ or $x-\dfrac{1}{x}=t$ as required.

$\int\dfrac{x^2+a^2}{x^4+kx^2+a^4}dx,\ \int\dfrac{x^2-a^2}{x^4+kx^2+a^4}dx,$ where k is a constant

These integrals can be obtained by dividing numerator and denominator by x^2, then putting

$x-\dfrac{a^2}{x}=t$ and $x+\dfrac{a^2}{x}=t$ respectively.

STANDARD SUBSTITUTIONS

(a) For terms of the form x^2+a^2 or $\sqrt{x^2+a^2}$, put $x=a\tan\theta$ or $a\cot\theta$ $\qquad\theta\in(0,\pi/2)$

(b) For terms of the form x^2-a^2 or $\sqrt{x^2-a^2}$, put $x=a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$ $\qquad\theta\in(0,\pi/2)$

(c) For terms of the form a^2-x^2 or $\sqrt{a^2-x^2}$, put $x=a\sin\theta$ or $a\cos\theta$ $\qquad\theta\in[0,\pi/2]$

(d) If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x=a\cos\theta$.

(e) For the type $\sqrt{(x-a)(b-x)}$, $\sqrt{\dfrac{x-a}{b-x}}$, put $x=a\cos^2\theta+b\sin^2\theta$ $\qquad\theta\in[0,\pi/2]$

(f) For the type $\sqrt{(x-a)(x-b)}$, $\sqrt{\dfrac{x-a}{x-b}}$, put $x=a\sec^2\theta-b\tan^2\theta$ $\qquad\theta\in(0,\pi/2)$

(g) For the type $\left(\sqrt{x^2+a^2}\pm x\right)^n$ or $\left(x\pm\sqrt{x^2-a^2}\right)^n$, put the expression within the bracket $=t$.

(h) For the type $(x+a)^{-1-\frac{1}{n}}(x+b)^{-1+\frac{1}{n}}$ or $\left(\frac{x+b}{x+a}\right)^{\frac{1}{n}-1}\frac{1}{(x+a)^2}$ $(n\in N,\ n>1)$, put $\frac{x+b}{x+a}=t.$

(i) For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1,\ n_2\in N$ (and >1), again put $(x+a)=t(x+b)$

Euler's Substitution

The integral of the form $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ are calculated with the aid of one of the three **Euler's substitution**.

(i) $\sqrt{ax^2+bx+c}=t\pm x\sqrt{a}$, If $a>0;$

(ii) $\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}$, If $c>0;$

(iii) $\sqrt{ax^2+bx+c}=(x-\alpha)t$, If $ax^2+bx+c=a(x-\alpha)(x-\beta),$

i.e., If α is a real root of the trinomial $ax^2+bx+c=0.$

INTEGRATION OF IRRATIONAL ALGEBRAIC FRACTIONS

1. Integrals of the form :

(i) $\int\frac{dx}{ax^2+bx+c}$ (ii) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ (iii) $\int\sqrt{ax^2+bx+c}\ dx$

Working Rule :

Write $ax^2+bx+c=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]=a\left[x^2+2x.\frac{b}{2a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right]=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$

Thus, ax^2+bx+c will be reduced to the form A^2+X^2 or A^2-X^2 or X^2-A^2, where X is a linear expression in x and A is a constant.

2. Integral of the form :

(i) $\int\frac{px+q}{ax^2+bx+c}dx$ (ii) $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

(iii) $\int(px+q)\sqrt{ax^2+bx+c}\ dx$ (iv) $\int\frac{px^2+qx+r}{ax^2+bx+c}dx$

Working Rule :

(a) In (i), (ii) and (iii), write $px+q=A$ [d.c. of (ax^2+bx+c)] $+B$ and find A and B equating the coefficient of similar powers of x and thus one part will be easily integrable and for other part proceed as in (1).

(b) In case (iv), write $px^2+qx+r=A(ax^2+bx+c)+B(2ax+b)+C$ and find $A,\ B,\ C.$

3. Integrals of the form $\int\frac{1}{X\sqrt{Y}}dx$ where X and Y are linear or quadratic expression in $x.$

Use the substitution as given in the following table and proceed.

X	Y	Substitution
Linear	Linear	$z^2 = Y$
Quadratic	Linear	$z^2 = Y$
Linear	Quadratic	$z = \frac{1}{X}$
Pure Quadratic	Pure Quadratic	$z^2 = \frac{Y}{X}$ or $z = \frac{1}{x}$

4. Integral of the form :

(i) $\int \left(x \pm \sqrt{a^2 + x^2}\right)^n dx$ (ii) $\int \frac{dx}{\left(x \pm \sqrt{a^2 + x^2}\right)^n}$ (iii) $\int \frac{\left(x \pm \sqrt{a^2 + x^2}\right)^n}{a^2 + x^2} dx$

where n is a positive rational number and $n \neq \pm 1$.

Working Rule : Put $z = x \pm \sqrt{a^2 + x^2}$.

5. Integrals of the form :

(i) $\int \frac{dx}{(x-a)^m (x-b)^n}$, where m and n are natural numbers and $a \neq b$.

Working Rule : Put $x - a = z(x - b)$

(ii) $\int R\left[x, (ax+b)^{\alpha/n}\right] dx,$ where R means for a rational functional.

Working Rule : Put $z^n = ax + b$

(iii) $\int R\left[x, (ax+b)^{\alpha/n}, (ax+b)^{\beta/m}\right] dx$

Working Rule : Put $z^p = ax + b,$ where $p =$ L.C.M. for m and n

(iv) $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{\alpha}{n}}\right] dx$

Working Rule : Put $z^n = \frac{ax+b}{cx+d}$.

6. Integrals of the form $\int x^m \left(a + bx^n\right)^p dx,$ Where m, n and p are rational numbers can be solved as follows :

(i) If $p \in N$, Expand the integral with the help of binomial theorem and then integrate.

(ii) If p is a negative integer, the integral reduces to the integral of a rational function by means of the substitution $x = t^s$, where s is the L.C.M. of denominators of the fractions m and n.

(iii) If $\frac{(m+1)}{n}$ is an integer, the integral can be rationalized by the substitution $a + bx^n = t^s$ where s is the denominator of the fraction P.

(iv) If $\frac{(m+1)}{n} + p$ is an integer, substitute $ax^{-n} + b = t^s$, where s is the denominator of the fraction P.

7. Integrals of the form $\int \frac{1}{(x-k)\sqrt{ax^2+bx+c}}\,dx$, the substitution $x-k=1/t$ reduces the integral $\int \frac{1}{(x-k)\sqrt{ax^2+bx+c}}\,dx$ to the problem of integrating an expression of the form $\frac{1}{\sqrt{At^2+Bt+C}}$.

8. $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+x}}$. Here we substitute, $x-k=1/t$.

9. Integrals of the form $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$ are solved by isolating in the numerator, the derivative of the quadratic appearing under the root sign and expanding the integral into the sum of two integrals.

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{(A/2a)(2ax+b)+B-(Ab/2a)}{\sqrt{ax^2+bx+c}}\,dx$$

$$= \frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(B-A\frac{b}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

10. Integrals of the form $\int \frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+........+a_{n-1}x+a_n}{\sqrt{ax^2+bx+c}}\,dx$ are solved as follows

Here, we assume that

$$\int \frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_{n-1}x+a_n}{\sqrt{ax^2+bx+c}}\,dx$$

$$=\left(C_0x^{n-1}+C_1x^{n-2}+.....+C_{n-1}\right)\sqrt{ax^2+bx+c}+C_n\int \frac{dx}{\sqrt{ax^2+bx+c}} \to \text{(I)}$$

Where $C_0, C_1, C_2, ..., C_n$ are arbitrary constants

Now differentiating both of (I) w.r.t x and multiplying by $\sqrt{ax^2+bx+c}$ we get.

$$a_0x^n+a_1x^{n-1}+.......a_{n-1}x+a_n$$

$$=\left[(n-1)C_0x^{n-2}+(n-2)C_1x^{n-3}+....+C_{n-2}\right]\left(ax^2+bx+c\right)+\frac{1}{2}\left(C_0x^{n-1}+C_1x^{n-2}+.......+C_{n-1}\right)(2ax+b)+C_n$$

Where constant $C_0, C_1, C_2, ..., C_n$ can be evaluate by comparity of like power of x four both sides.

Substituting the values of $C_0, C_1, C_2......C_n$ in (I) and evaluating $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ the given integral is determined completely.

11. $\int \frac{\left(ax^2+bx+c\right)dx}{(dx+e)\sqrt{fx^2+gx+h}}$.

Here, we write, $ax^2+bx+c=A_1(dx+e)(2fx+g)+B_1(dx+e)+C_1$ where A_1, B_1 and C_1 are constants which can be obtained by comparing the coefficient of like terms on both sides. And given integral will reduce to the form

$$A_1\int \frac{(2fx+g)}{\sqrt{fx^2+gx+h}}\,dx+B_1\int \frac{dx}{\sqrt{fx^2+gx+h}}+C_1\int \frac{dx}{(dx+e)\sqrt{fx^2+gx+h}}$$

INTEGRALS OF THE FORM $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$ **and** $\int \frac{dx}{a+b\sin x+c\cos x}$

To evaluate such form proceed as follows :

1. Put $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

2. Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$.

3. Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$.

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1. Integral of the form $\int \sin^m x\cos^n x\,dx$:

   (i) To evaluate the integrals of the form $I = \int \sin^m x\cos^n x\,dx$, where m and n are rational numbers.

   (a) Substitute $\sin x = t$, if n is odd ;

   (b) Substitute $\cos x = t$, if m is odd ;

   (c) Substitute $\tan x = t$, if m + n is a negative even integer; and

   The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$ However, in practice; it sometimes leads to very complex rational function. In some cases, the integral can be simplified by :

   (a) Substitute $\sin x = t$, if the integral is of the form $\int R(\sin x)\cos x\,dx$.

   (b) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x)\sin x\,dx$.

   (c) Substituting $\tan x = t$, *i.e.* $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on tan x.

   (d) If the given function is $R(\sin x+\cos x)\cos 2x$

   Put $\sin x+\cos x = t \Rightarrow 1+\sin 2x = t^2 \Rightarrow \cancel{2}\cos 2x\,dx = \cancel{2}t\,dt$

   And if function is $R(\sin x-\cos x)\cos 2x$

   Put $\sin x-\cos x = t \Rightarrow 1-\sin 2x = t^2 \Rightarrow -\cancel{2}\cos 2x\,dx = \cancel{2}t\,dt$ and proceed further.

INTEGRALS OF THE FORM :

(i) $\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}dx$

In this integral express numerator as l (Denominator) + m (d.c. of denominator) + n.

Find l, m, n by comparing the coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals.

$$l\int dx + m\int \frac{\text{d.c. of (Denominator)}}{\text{Denominator}}dx + n\int \frac{dx}{a\cos x+b\sin x}$$

(ii) $\int \frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx$

Express numerator as l (denominator) + m (d.c. of denominator) and find l and m by comparing the coefficients of $\sin x$, $\cos x$.

Definite Integral Chapter 22

FIRST FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $F(x)$ is one of the antiderivatives of a function $f(x)$ continuous on $[a, b]$, then :

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

The following formula due to **Newton** and **Leibnitz** therefore this is also called **Newton Leibnitz formula**.

PROPERTIES OF DEFINITE INTEGRALS

1. $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)\,dx$

2. $\int\limits_a^b f(x)dx = \int\limits_a^b f(y)dy$

3. $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)\,dx,$ where $a < c < b$

4. $\int\limits_o^{2a} f(x)dx = \int\limits_o^a \left[f(x) + f(2a - x)\right]dx$

   Corollary : $\int\limits_o^{2a} f(x)dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int\limits_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$

5. $\int\limits_a^b f(x)dx = \int\limits_a^b f(a+b-x)dx$

   Corollary : $\int\limits_o^a f(x)dx = \int\limits_o^a f(a-x)dx$

6. $\int\limits_{-a}^{a} f(x)dx = \int_0^a \left[f(x) + f(-x)\right]dx$

   $= \begin{cases} 2\int\limits_0^a f(x)dx, & \text{if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

7. $\int\limits_o^a \dfrac{f(x)}{f(x) + f(a-x)}dx = \dfrac{a}{2}$

8. $\int\limits_a^b \dfrac{f(x)}{f(x) + f(a+b-x)}dx = \dfrac{b-a}{2}$

9. If $f(x)$ is a periodic function of period T i.e. $f(T+x) = f(x)$, then

   (a) $\int\limits_o^{nT} f(x)dx = n\int\limits_o^T f(x)dx$

   (b) $\int\limits_{mT}^{nT} f(x)dx = (n-m)\int\limits_o^T f(x)dx$

(c) $\int_a^b f(x)\,dx = \int_{a+nT}^{b+nT} f(x)\,dx$ (d) $\int_a^{a+nT} f(x)\,dx$ is independent of a.

10. $\int_a^b f(x)\,dx = (b-a)\int_0^1 f((b-a)x+a)\,dx$

11. If $f(x) \ge 0$ on the interval $[a, b]$, then $\int_a^b f(x) \ge 0.$

12. If a function f is integrable and non-negative on $[a, b]$ and there exists a point $c \in [a, b]$ of continuity of f for which $f(c) > 0,$ then $\int_a^b f(x)\,dx > 0 \quad (a < b).$

13. If $f(x) \le g(x)$ on the interval $[a, b],$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

14. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

15. If $f(x)$ is continuous on $[a, b],$ m is the least and M is the greatest value of $f(x)$ on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

16. **Schwarz – Bunyakovsky inequality :**
For any two functions $f(x)$ and $g(x),$ integrable on the interval $[a, b]$, the **Schwarz – Bunyakovsky** inequality holds i.e. $\left|\int_a^b f(x)\cdot g(x)\,dx\right| \le \sqrt{\int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx}$

17. If a function $f(x)$ is continuous on the interval $[a, b],$ then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a), \text{ where } a < c < b.$$

(a) **Mean Value or Average Value of a function :**
Mean Value or Average Value of a function on the interval $[a, b]$ is given by

$$f(x) > = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

(b) **Root Mean Square Value (RMSV) :**
Root Mean Square Value (RMSV) of a function $y = f(x)$ on the interval (a, b) is

$$\sqrt{\left(\frac{\int_a^b y^2 dx}{(b-a)}\right)}$$

18. **Holder's Inequality for Integrals**

$$\int_a^b |f(x)g(x)|\,dx \le \left\{\int_a^b |f(x)|^p\,dx\right\}^{\frac{1}{p}} \left\{\int_a^b |g(x)|^q\,dx\right\}^{\frac{1}{q}} \text{ where } \frac{1}{p}+\frac{1}{q}=1,\ p>1,\ q>1.$$

If $p = q = 2$, this reduces to Cauchy-Schwarz Inequality for integrals.

The equality holds if and only if $\dfrac{|f(x)|^{p-1}}{|g(x)|}$ is a constant.

19. **Minkowski's Inequality for Integrals**

If $p>1$, $\left\{\int_a^b |f(x)+g(x)|^p\,dx\right\}^{\frac{1}{p}} \le \left\{\int_a^b |f(x)|^p\,dx\right\}^{\frac{1}{p}} + \left\{\int_a^b |q(x)|^p\,dx\right\}^{\frac{1}{p}}$

The equality holds if and only if $\dfrac{f(x)}{g(x)}$ is a constant.

SECOND FUNDAMENTAL THEOREM

If f is continuous on $[a, b]$ then

$$F(x)=\int_a^x f(t)\,dt$$

is differentiable at every point x in $[a, b]$ and $\dfrac{dF}{dx}=\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x).$

Leibnitz's rule

1. If $f(x)$ is continuous on $[a, b]$, and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\{v(x)\}\frac{d}{dx}\{v(x)\} - f\{u(x)\}\frac{d}{dx}\{u(x)\}.$$

2. If the function $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at a point $x\in(a,b)$, and $f(x,t)$ is continuous then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right]=\int_{\phi(x)}^{\psi(x)}\frac{\partial}{\partial x} f(x,t)\,dt+\left\{\frac{d\psi(x)}{dx}\right\} f(x,\psi(x))-\left\{\frac{d\phi(x)}{dx}\right\} f(x,\phi(x)).$$

SUMMATION OF SERIES WITH THE HELP OF DEFINITE INTEGRAL AS THE LIMIT OF A SUM :

If $f(x)$ is a continuous and single valued function defined on the interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ is defined as follows :

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a+h)+f(a+2h)+...+f(a+nh)\right] \text{ where } nh=b-a.$$

or $\lim\limits_{\substack{n\to\infty\\ h\to 0}} h\sum\limits_{r=1}^{n} f(a+rh)=\int_a^b f(x)\,dx$ … (1)

Put $a=0$ and $b=1 \Rightarrow nh=1 \Rightarrow h=\dfrac{1}{n}.$

Substitute $a=0, b=1$ and $h=\dfrac{1}{n}$ in (1), we get $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{r=1}^{n} f\left(\dfrac{r}{n}\right)=\int_0^1 f(x)\,dx.$

Also $\int_a^b f(x)\,dx=\lim\limits_{r\to 1} a(r-1)\left[f(a)+rf(ar)+......r^{n-1}f(ar^{n-1})\right]$

IMPROPER INTEGRALS :

1. $\int_a^{\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx.$

2. $\int_{-\infty}^{b} f(x)\,dt = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$

3. $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx$

BETA AND GAMMA FUNCTIONS

Definitions:

1. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad [m>0, n>0]$

   is called the **First Eulerian Integral** or **Beta function**.

2. $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx \quad [n>0]$

   is called the **Second Eulerian Integral** or **Gamma function**.

PROPERTIES OF BETA AND GAMMA FUNCTIONS

(a) $\int_0^{\infty} e^{-x}x^n dx = n\int_0^{\infty} e^{-x}x^{n-1}dx$

$\therefore \quad \Gamma(n+1) = n\Gamma(n).$

when n is a positive integer, $\qquad \Gamma(n+1) = n!$

(b) $\int_0^{\infty} e^{-kx}x^{n-1}dx = \dfrac{\Gamma(n)}{k^n}, \qquad [k>0, n>0]$

(c) $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \qquad [m>0,\ n>0]$

(d) $\Gamma(m)\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$ where $(0<m<1)$.

Corollary : putting $m = \dfrac{1}{2}$

we get $\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2}\right) = \dfrac{\pi}{\sin\frac{1}{2}\pi} = \pi \qquad \therefore \quad \Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}.$

(e) $B(m,n) = \int_0^{\infty}\dfrac{x^{m-1}dx}{(1+x)^{m+n}} = \int_0^{\infty}\dfrac{x^{n-1}dx}{(1+x)^{m+n}}.$

(f) $\int_0^{\frac{1}{2}\pi}\sin^p\theta\cos^q\theta\,d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \qquad [p>-1,\ q>-1]$

Walli's Formula

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}...\dfrac{2}{3}, & \text{when n is odd} \\ \dfrac{n-1}{n}.\dfrac{n-3}{n-2}.\dfrac{n-5}{n-4}...\dfrac{3}{4}.\dfrac{1}{2}.\dfrac{\pi}{2}, & \text{when n is even} \end{cases}$$

$$\int_0^{\pi/2} \sin^m x \cos^n dx = \frac{\{(m-1)(m-3)...2 \text{ or } 1\}\{(n-1)(n-3)...2 \text{ or } 1\}}{(m+n)(m+n-2)...(2 \, or \, 1)},$$

[If *m*, *n* are both odd positive integers or one odd positive integer]

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\{(m-1)(m-3)...1\}\{(n-1)(n-3)...1\}}{(m+n)(m+n-2)...2} \cdot \frac{\pi}{2}$$

[If *m*, *n* are both even positive integers]

PARTICULAR CASES

1. In case any of ***m* or *n* is 1**.

(a) $\int_0^{\pi/2} \sin^m x \cos x dx = \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$

(b) $\int_0^{\pi/2} \cos^m x \sin x dx = -\left[\frac{\cos^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}$

2. If any of ***m* or *n* is zero** we put 1 as the only factor in its product and we regard 0 as even.
3. When limits are 0 to π

$\int_0^{\pi} \sin^m x \cos^n x dx = 0$ always except when n the power of $\cos x$ is even.

CURVE TRACING

In order to find the area bounded by several curves, it is important to have rough sketch of the required portion. The following steps are very useful in tracing a cartesian curve $f(x, y) = 0$.

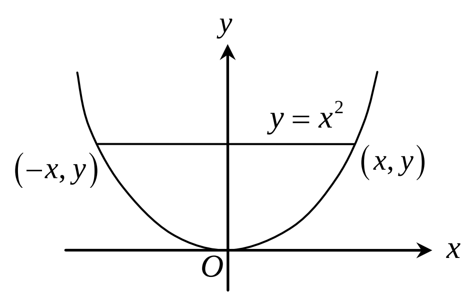
1. **SYMMETRY**

Certain kinds of symmetries in the graph of the equation $F(x, y) = 0$ can easily be detected in the following way.

Symmetry about the *y-axis* : The graph (curve) is symmetric about the y-axis if the equation is unaltered when x is replaced by $-x$; that is, if

$$F(x, y) = F(-x, y),$$

then the points (x, y) and $(-x, y)$ both lie on the curve (satisfy the equation $F = 0$) if either one of them does. In particular, an equation that contains only even powers of x represents such a curve



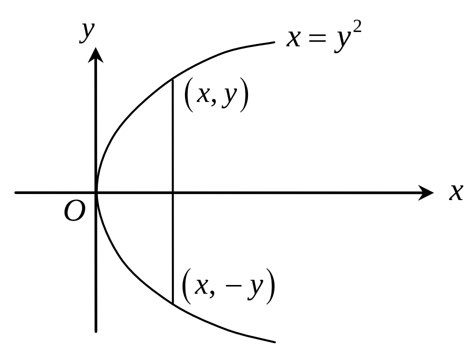
Symmetry about the y-axis

Symmetry about the *x-axis* :

The graph is symmetric about the x-axis if the equation is unaltered when y is replaced by –*y*; that is, if

$$F(x, y) = F(x, -y).$$

In particular, this will happen if only even powers of *y* occur in the equation.

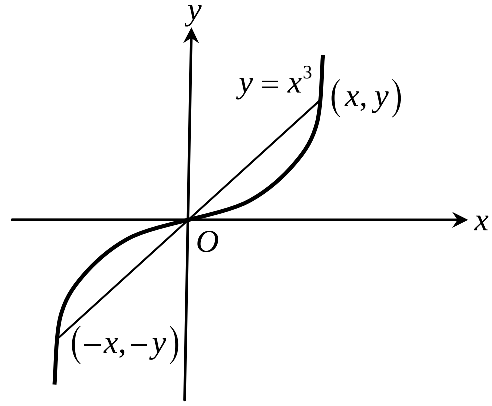


Symmetry about the x-axis

Symmetry about the origin :

The equation is unaltered when x and y are replaced by $-x$ and $-y$; that is,

$$F(x,y)=F(-x,-y).$$

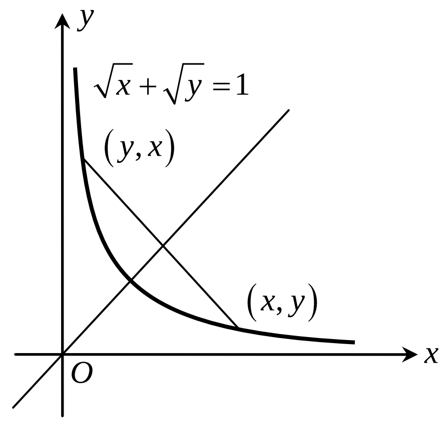


Symmetry about the origin

Symmetry about the line $y=x$

The equation is unaltered when x and y are interchanged, that is,

$$F(x,y)=F(y,x).$$

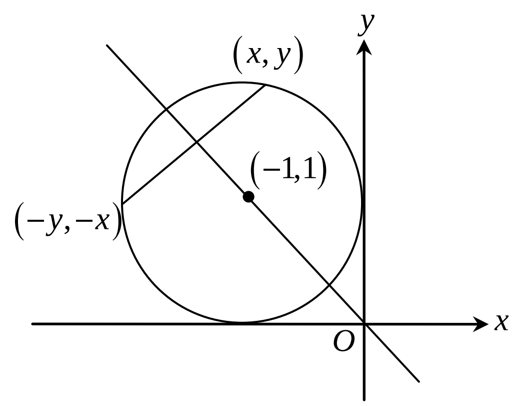


Symmetry about the line $y=x$

Symmetry about the line y = – x

The equation is unaltered when x and y are replaced by –y and –x respectively; that is,

$$F(x,y)=F(-y,-x).$$



Symmetry about the line $y=-x$

Example 1

The graph of the equation $x^2+y^2=1$ is symmetric about both axes, the origin, the line $y=x$, and the line $y=-x$

Example 2

The graph of the equation $x^2-y^2=1$ is symmetric about both axes and the origin and not symmetric about the lines $y = x$ and $y = -x$.

Example 3

The graph of the equation $xy=1$ is not symmetric about either axis but is symmetric about the origin and about the lines $y = x$, and $y = -x$.

2. **ORIGIN**

If there is no constant term in the equation of the given curve, then the curve passes through the origin.

In that case, the given equation is a polynomial equation in x and y then the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the given curve.

For example, the curve $y^3 + x^3 + axy = 0$ passes through the origin and the tangents at the origin are given by $axy = 0$ i.e. $x = 0$ and $y = 0$.

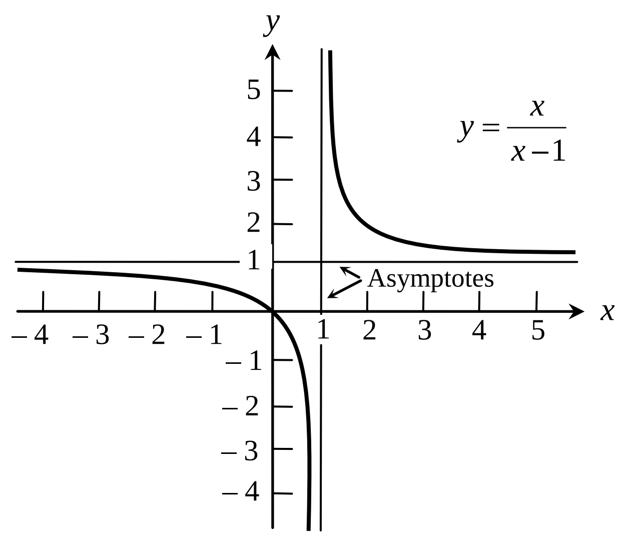
3. **INTERSECTION WITH THE CO-ORDINATES AXES**

(i) To find the points of intersection of the curve with X-axis, put $y = 0$ in the equation of the given curve and get the corresponding values of x.

(ii) To find the points of intersection of the curve with Y-axis, put $x = 0$ in the equation of the given curve and get the corresponding values of y.

4. **ASYMPTOTES**

As a point P on the graph of a function $y = f(x)$ moves farther and farther away from the origin, it may happen that the distance between P and some fixed line tends to zero. In other words, the curve "approaches" the line as it gets further from the origin. In such a case, the line is called an ***asymptote*** of the graph. For instance, the x-axis and y-axis are asymptotes of the curves $y = 1/x$ and $y = 1/x^2$.



The graph of $y = x/(x-1)$.

To find out the asymptotes of the curve if the equation is a polynomial equation in x and y.

(i) The vertical asymptotes or the asymptotes parallel to y-axis of the given curve are obtained by equating to zero the coefficient of the highest power of y in the equation of the given curve.

(ii) The horizontal asymptotes or the asymptotes parallel to x-axis of the given curve are obtained by equating to zero the coefficient of the highest power of x in the equation of the given curve.

Horizontal and vertical Asymptotes

A line y = b is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to\infty} f(X) = b$$

A line $x = a$ is a vertical asymptote of the graph if either.

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

5. **REGION**

Find out the regions of the plane in which no part of the curve lies. To determine such regions we solve the given equation for y in terms of x or vice-versa. Suppose that y becomes imaginary for $x > a$, the curve does not lie in the region $x > a$.

6. **CRITICAL POINTS**

These are the points at which either the derivative of the function is zero or it does not exist.

Find out the values of x at which $\frac{dy}{dx}=0.$

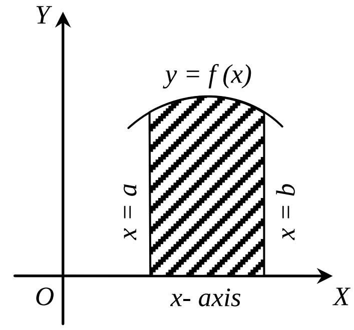
At such points y generally changes its character from an increasing function of x to a decreasing function of x or vice-versa.

7. Trace the curve with the help of the above points.

AREA OF PLANE REGIONS

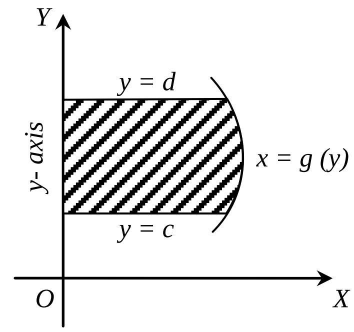
1. The area bounded by the curve $y=f(x),$ x-axis and the ordinates $x=a.$ and $x=b$ (where $b>a$) is given by

$$A=\int_a^b y\ dx=\int_a^b f(x)\ dx$$



2. The area bounded by the curve $x=g(y),$ y-axis and the abscissae $y=c$ and $y=d$ (where $d>c$) is given by

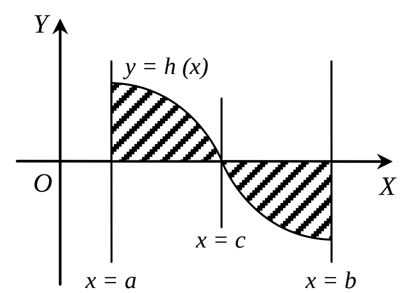
$$A=\int_c^d x\ dy=\int_c^d g(y)\ dy.$$



3. The area bounded by the curve $y=h(x),$ x-axis and the two ordinates $x=a$ and $x=b$ is given by

$$A=\left|\int_a^c y\ dx\right|+\left|\int_c^b y\,dx\right|$$

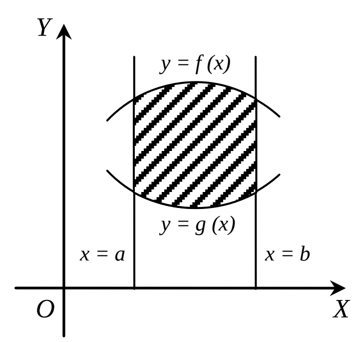
where c is a point in between a and $b.$



4. If we have two curve $y=f(x)$ and $y=g(x)$, such that $y=f(x)$ lies above the curve $y=g(x)$ then the area bounded between them and the ordinates $x=a$ and $x=b(b>a),$ is given by

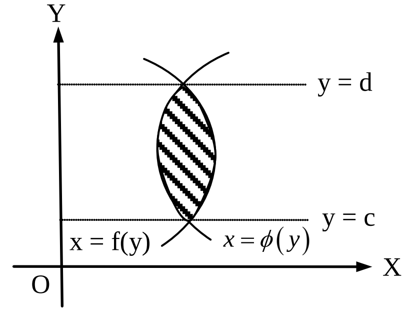
$$A=\int_a^b\left[f(x)-g(x)\right]dx$$

i.e. upper curve area $-$ lower curve area.



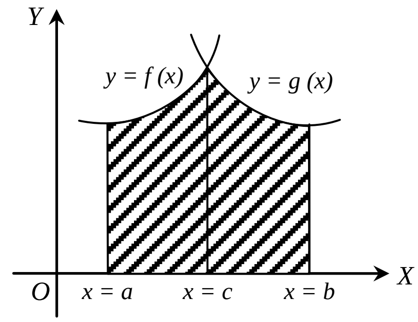
5. The area bounded between the curve $x = f(y), x = \phi(y)$ and abscissae $y = c$ and $y = d$ is given by

$$A = \int_c^d \{f(y) - \phi(y)\} dy$$



6. The area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$ is given by where $x = c$ is the point of intersection of the two curves is given by

$$A = \int_a^c f(x) dx + \int_c^b g(x) dx.$$



PARAMETRIC FORM FOR THE AREA UNDER A CURVE

Let the curve be given by $x = \phi(t);\ y = \Psi(t)$. If, on eliminating t, y can be expressed as a function of x (or x as a function of y), then the area in equation can be obtained by the formula.

If, however, the parameter can not be eliminated easily, then we use the formula :

$$\text{Area} = \int_a^b y\, dx = \int_{t_1}^{t_2} \Psi(t) \varphi'(t) dt$$

where t_1 and t_2 are given by $a = \varphi(t_1)$ and $b = \varphi(t_2)$.

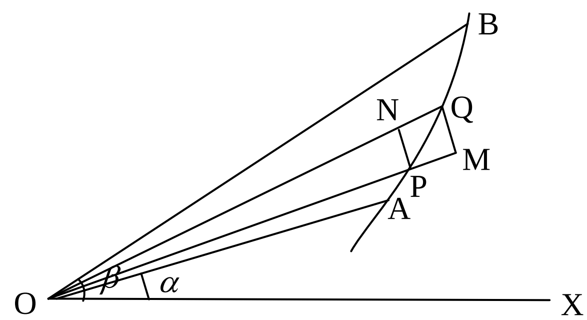
AREA IN POLAR CO-ORDINATES

Let $r = f(\theta)$ be a curve APB, where $f(\theta)$ is supposed to be a finite, continuous and single valued function in the interval $\alpha < \theta < \beta$. The area bounded by the curve, and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

or

$$A = \frac{1}{2}\int_\alpha^\beta \{f(\theta)\}^2 d\theta.$$



Differential Equation **Chapter 23**

DIFFERENTIAL EQUATION

An equation involving an independent variable, a dependent variable and the derivatives of the dependent variable and their power are called *differential equation*.

ORDER OF A DIFFERENTIAL EQUATION

The order of highest order derivative appearing in a differential equation is called the *order* of the differential equation.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives.

FORMATION OF A DIFFERENTIAL EQUATION

An equation involving-independent variable x, the dependent variable y and 'n' independent arbitrary constants, to form the differential equation of such family of curves we have to eliminate the 'n' independent arbitrary constants form the given equation.

This can be achieved by differentiating given equation n times and, we get a differential equation of nth order corresponding to given family of curves.

SOLUTION OF A DIFFERENTIAL EQUATION

Any relation between the dependent and independent variables (not involving the derivative) which, when substituted in the differential equation reduces it to an identity is called a *solution* of the differential equation.

GENERAL SOLUTION

The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the *general solution*. Thus, the general solution of a differential equation of the nth order has n arbitrary constants.

PARTICULAR SOLUTION

A solution obtained by giving particular values to arbitrary constant in the general solution is called a *particular solution*.

SINGULAR SOLUTION

This solution does not contain any arbitrary constant nor can it be derived from the complete solution by giving any particular value to the arbitrary constant, it is called the *singular solution* of the differential equation.

The singular solution represents the envelope of the family of straight lines represented by the complete solution.

SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

The following methods may be used to solve first order and first degree differential equations.

1. **VARIABLE SEPARABLE DIFFERENTIABLE EQUATIONS**

A differential equation of the form $f(x)+g(y)\frac{dy}{dx}=0$ …(1)

or $f(x)dx+g(y)dy=0$, is said to have separated variables.

Integrating (1), we obtain $\int f(x)\,dx+\int g(y)\frac{dy}{dx}dx=C$, where C is an arbitrary constant.

Hence, $\int f(x)\,dx+\int g(y)\,dy=C$ is the solution of (1).

2. **EQUATIONS REDUCIBLE TO SEPARABLE FORM**

Sometimes in a given differential equation, the variables are not separable. But, some suitable substitution reduces to a form in which the variables are separable *i.e.*, the differential equation of the type $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to variable separable form by substitution $ax+by+c=t$. The reduced variable separable form is :

$$\frac{dt}{bf(t)+a} = dx .$$

Integrate both sides to obtain the solution of this differential equation.

3. **HOMOGENEOUS DIFFERENTIAL EQUATION**

A differential equation $P(x,y)dx + Q(x,y)dy = 0$ ....(*)

is called **homogenous,** if $P(x,y)$ *and* $Q(x,y)$ are homogenous functions of the same degree in x *and* y. Equation (*) may be reduced to the from $y' = f\left(\frac{y}{x}\right)$. By means of the substitution $y = vx$, where v is some unknown function, the equation is transformed to an equation with variable separable. (We can also use substitution $x = vy$).

4. **EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM**

TYPE I: Consider a differential equation of the form :

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \text{ where } \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \qquad \ldots(1)$$

This is clearly non-homogeneous. In order to make it homogeneous, we proceed as follows :

We substitute $x = X + h$ and $y = Y + k$ in (1), where h, k are constants to be determined suitably.

We have $\frac{dx}{dX} = 1$ and $\frac{dy}{dY} = 1$, so that

$$\frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}.$$

Now (1) becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \qquad \ldots(2)$$

Choose h and k so that

$$a_1h + b_1k + c_1 = 0,$$

$$a_2h + b_2k + c_2 = 0.$$

These equations give

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\ k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \qquad \ldots(3)$$

Now equation (2) becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y},$$

which being a homogeneous equation can be solved by means of the substitution $Y = VX$.

TYPE II : Consider a differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \text{ where } \frac{a_1}{a_2} = \frac{b_1}{b_2} = k(\text{say})$$

Since $a_1b_2 - a_2b_1 = 0$, the above method fails in view of (3).

We have $$\frac{dy}{dx} = \frac{k\left(a_2x + b_2y\right) + c_1}{a_2x + b_2y + c_2} \qquad \ldots(4)$$

Substituting $a_2x + b_2y = z$ so that $a_2 + b_2\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Now (4) becomes $\dfrac{dz}{dx} = b_2 \cdot \dfrac{kz + c_1}{z + c_2} + a_2$, which is an equation with variables separable.

EXACT DIFFERENTIAL EQUATION

If *M* and *N* are functions of *x* and *y*, the equation $Mdx + Ndy = 0$ is called exact when there exists a functions, $f\left(x, y\right)$ of *x* and *y* such that

$$d\left[f\left(x, y\right)\right] = Mdx + Ndy \ \ i.e. \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

where $\dfrac{\partial f}{\partial x} =$ Partial derivative of $f\left(x, y\right)$ with respect to x (keeping *y* constant).

$\dfrac{\partial f}{\partial y} =$ Partial derivative of $f\left(x, y\right)$ with respect to *y* (treating *x* as constant).

The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0 \qquad \text{to be exact is} \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

An exact differential equation can always be derived from its general solution directly by differentiation without any subsequent multiplication elimination *etc*.

Integrating factor

If an equation of the form $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying by some function of *x* and *y*. Such a multiplier is called an *integrating factor*.

Working rule for solving an exact differential equation

Step (i) Compare the given equation with $Mdx + Ndy = 0$ and find out *M* and *N*. Then find out $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$. If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ the given equation is exact.

Step (ii) Integrate *M* with respect to *x* treating *y* as a constant.

Step (iii) Integrate *N* with respect to *y* treating *x* as constant and omit those terms which have been already obtained by integrating *M*.

Step (iv) On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution. In other words, solution of an exact differential equation is

$$\int_{\substack{\text{Regarding } y \text{ as} \\ \text{constant}}} Mdx + \int_{\substack{\text{Only those terms} \\ \text{not containing } x}} Ndy = c$$

METHODS FOR SOLVING EXACT DIFFERENTIAL EQUATIONS

1. **Solution by inspection**

If we can write the differential equation in the form

$$f\left(f_1\left(x, y\right) df_1\left(x, y\right)\right) + \phi\left(f_2\left(x, y\right) d\left(f_2\left(x, y\right)\right)\right) + \ldots = 0,$$

then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d\left(x + y\right) = dx + dy$ (ii) $d\left(xy\right) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right)=\frac{2xy\,dx-x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-xdy}{x^2+y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$

(xi) $d\left[\ln(xy)\right]=\frac{xdy+ydx}{xy}$

(xii) $d\left(\ln\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\ln\left(x^2+y^2\right)\right]=\frac{xdx+ydy}{x^2+y^2}$

(xiv) $d\left[\ln\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xvii) $d\left(\frac{e^v}{x}\right)=\frac{xe^vdy-e^ydx}{x^2}$

(xviii) $d\left(x^m y^n\right)=x^{m-1}\,y^{n-1}\left(my\,dx+nx\,dy\right)$

(xix) $d\left(\sqrt{x^2+y^2}\right)=\frac{xdx+ydy}{\sqrt{x^2+y^2}}$

2. If the given equation $Mdx+Ndy=0$ is homogeneous and $\left(Mx+Ny\right)\neq 0$, then $1/\left(Mx+Ny\right)$ is an integrating factor.
3. If the equation $Mdx+Ndy=0$ is of the form $f_1\left(xy\right)y\;dx+f_2\left(xy\right)x\;dy=0$, then $1/\left(Mx-Ny\right)$ is an integrating factor of $Mdx+Ndy=0$ provided $\left(Mx-Ny\right)\neq 0$.
4. If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone say $f\left(x\right)$, then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.
5. If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is function of y alone say $f\left(y\right)$, then $e^{\int f(y)dy}$ is an integrating factor of $Mdx+Ndy=0$.
6. If the given equation $Mdx+Nay=0$ is of the form $x^{\alpha}y^{\beta}\;\left(my\;dx+nx\,dy\right)=0$, then its I.F. is $x^{km-1-\alpha}y^{kn-1-\beta}$, where k can have any value.

LINEAR EQUATION

An equation of the form $\frac{dy}{dx}+Py=Q$

In which P and Q are functions of x alone or constant is called a linear equation of the first order. The general solution of the above equation can be found as follows.

Multiply both sides of the equation by $e^{\int Pdx}$.

$$\therefore \qquad \frac{dy}{dx}e^{\int Pdx}+Pye^{\int Pdx}=Qe^{\int Pdx}$$

i.e., $\frac{d}{dx}\left(ye^{\int P\,dx}\right)=Qe^{\int P\,dx}.$

$\therefore$ integrating $ye^{\int P\,dx}=\int Qe^{\int P\,dx}dx+c$

or $\mathrm{y}=e^{-\int \mathrm{P}\,dx}\left[\int Qe^{\int \mathrm{P}\,dx}\mathrm{dx}+\mathrm{c}\right]$ is the required solution.

Cor.1 If in the above equation if Q is zero, the general solution is $y = Ce^{-\int P\,dx}$.

Cor.2 If P be a constant and equal to $-m$, then the solution is $y = e^{mx}\left[\int e^{-mx}Q\,dx+C\right]$.

Linear differential equations of the form $\frac{dx}{dy}+Rx=s.$

Sometimes a linear differential equation can be put in the form

$$\frac{dx}{dy}+Rx=s$$

where *R* and *S* are functions of *y* alone or constants.

Note : y is independent variable and x is a dependent variable.

Bernoulli's Equation

An equation of the form $\frac{dy}{dx}+Py=Qy^n,$

Where P and Q are functions of x alone, is known as Bernoulli's equation. It is easily reduced to the linear form.

Dividing both sides by y^n, we get $y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q.$

Putting $y^{-n+1}=v,$ and hence $(-n+1)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$ the equation reduces to $\frac{dv}{dx}+(1-n)Pv=(1-n)Q.$

This being linear in v can be solved by the method of the previous article.

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curves at right angle is called an *orthogonal trajectory* of the family. *For example*, each straight line $y=mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2+y^2=a^2$.

Procedure for finding the orthogonal trajectory

(i) Let $f(x,\ y,\ c)=0$ be the equation, where *c* is an arbitrary parameter.

(ii) Differentiate the given equation w.r.t. *x* and then eliminate *c*.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation in (iii).

DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

The typical equation of the first order and the n^{th} degree can be written as

$$P^n+P_1p^{n-1}+P_2p^{n-2}+......+P_{n-1}p+P_n=0 \quad \dots \text{(i)}$$

where *p* stands for $\frac{dy}{dx}$ and $P_1,\ P_2,.....,P_n$ are function of *x* and *y*.

The complete solution of such an equation would involve only one arbitrary constant.

The equations which are of first order but not of the first degree, the following types of equations are discussed.

(a) Equations solvable for $p = dy/dx$
(b) Equations solvable for y
(c) Equations solvable for x
(d) Clairut's equations

(i) Resolving left side of equation (i) into factors we have

$$\left(p-f_1(x,y)\right)\left(p-f_2(x,y)\right)......\left(p-f_n(x,y)\right)=0 \quad \text{… (ii)}$$

It is evident from above that a solution of any one of the equations.

$$p-f_1(x,y)=0,\ p-f_2(x,y)=0,\ ...,\ p-f_n(x,\ y)=0 \quad \text{… (iii)}$$

is also a solution of (i).

Let the solution of equation (iii) be $g_1(x,y,c_1)=0,\ g_2(x,y,c_2)=0,.....g_n(x,y,c_n)=0$

Where $c_1, c_2, ...c_n$ are arbitrary constant of the integration.

These solutions are evidently just as general, if $c_1 = c_2 = ... = c_n$, since the individual solutions are all independent of one another and all the $c's$ can have any one of an infinite number of values. All the solution can thus without loss of generality be obtained into the single equation.

(ii) **Equation solvable for y**

Suppose the equation is put in the from $y = f(x, p)$

Differentiating this w.r.t x, we shall get an equation in two variables x and p; suppose the solution of the latter equation is $\phi(x, p, c) = 0$.

The p eliminated between this equation and original equation gives a equation between x and y and c, which is the required solution.

(iii) **Equation solvable for x**

Differentiating this w.r.t y and noting that $\dfrac{dx}{dy}=\dfrac{1}{p}$, we shall get an equation in two variables y and p. If p be eliminated between the solution of the latter equation (which contains an arbitrary constant) and the original equation we shall get the required solution.

(iv) **Clairut's equation :** The equation of the form $y = px + f(p)$ is known as **Clairut's equation**.

$$y = px + f(p) \quad \text{…(1)}$$

Differentiating w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0 \qquad \Rightarrow \quad \frac{dp}{dx} = 0, \quad \text{or} \quad x + f'(p) = 0$$

If $\dfrac{dp}{dx} = 0$, we have $p = \text{constant} = c$ (say).

Eliminating p from (1) we have $y = cx + f(c)$ as a solution.

If $x + f'(p) = 0$, then by eliminating p, we will obtain another solution. This solution is called *singular solution*.

Note : Sometime transformation to the polar co-ordinates facilitates separation of variables. In this connections it is convenient to remember the following differentials.

If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$

(ii) $x\,dy - y\,dx = r^2 d\theta$

(iii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

If $x = r\sec\theta$; $y = r\tan\theta$ then,

(i) $x\ dx - y\ dy = r\,dr$

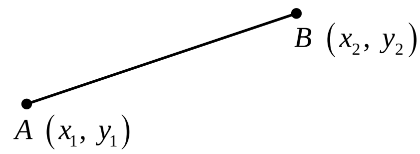
(ii) $x\ dy - y\ dx = r^2 \sec\theta\, d\theta$.

Straight line Chapter 24

DISTANCE FORMULA

The distance between two points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ is given by

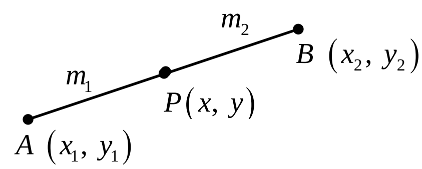
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



SECTION FORMULA

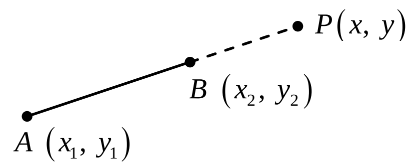
1. The coordinates of the point $P(x,\ y)$, dividing the line segment joining the two points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ internally in the ratio $m_1 : m_2$, are given by

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2},\ y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \qquad \frac{AP}{BP} = \frac{m_1}{m_2}$$



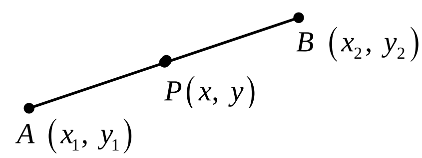
2. The coordinates of the point $P(x,\ y)$, dividing the line segment joining the two points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ externally in the ratio $m_1 : m_2$, are given by

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2},\ y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \qquad \frac{AP}{BP} = \frac{m_1}{m_2}$$



3. The coordinates of the mid point of the line segment joining the two points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ are given by

$$\left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right).$$

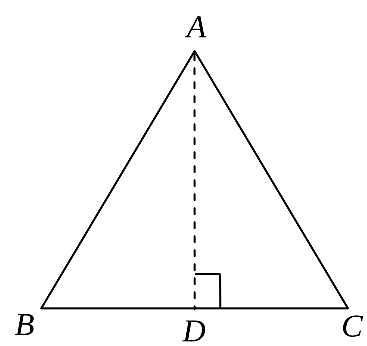


AREA OF A TRIANGLE

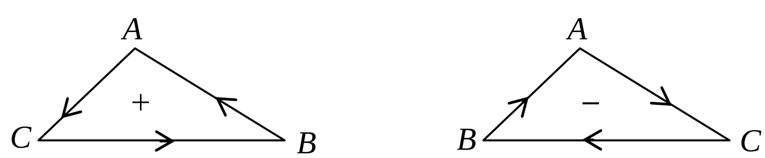
Let $(x_1,\ y_1), (x_2,\ y_2)$ and $(x_3,\ y_3)$ respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the are of triangle ABC, is

$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \qquad \ldots (1)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \ldots (2)$$



While using formula (1) or (2), order of the points $(x_1,\ y_1), (x_2,\ y_2)$ and $(x_3,\ y_3)$ has not been taken into account. If we plot the points $A(x_1,\ y_1), B(x_2,\ y_2)$ and $C(x_3,\ y_3)$, then the area of the triangle as obtained by using formula (1) or (2) will be positive or negative as the points A, B, C are in anti-clockwise or clockwise directions,



So, while finding the area of triangle ABC, we use the formula :

$$\text{Area of } \Delta ABC = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| = \frac{1}{2}\left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

Note :

(a) If the three points A, B, C are collinear then area of $\Delta\ ABC$ is zero.

(b) **The area of a quadrilateral**, whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, is

$$= \frac{1}{2}\left| \left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix} \right] \right|$$

(c) **The area of a polygon** of n sides with vertices $A_1(x_1, y_1)$, $A_2(x_2, y_2)$,...., $A_n(x_n, y_n)$ is

$$= \frac{1}{2}\left| \left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \dots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix} \right] \right|$$

(d) If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the equations of the sides of a triangle, then the area of the triangle is $= \dfrac{1}{2|C_1C_2C_3|} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$

where C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant

i.e. $C_1 = a_2b_3 - a_3b_2$, $C_2 = a_3b_1 - a_1b_3$ and $C_3 = a_1b_2 - a_2b_1$.

LOCUS

When a point moves in a plane under certain geometrical conditions, the point traces out a path. This path of a moving point is called its locus.

EQUATION OF LOCUS

The equation to a locus is the relation which exists between the coordinates of any point on the path and which holds for no other point except those lying on the path.

PROCEDURE FOR FINDING THE EQUATION OF THE LOCUS OF A POINT

(i) If we are finding the equation of the locus of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given conditions as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminate contains only h, k and known quantities.

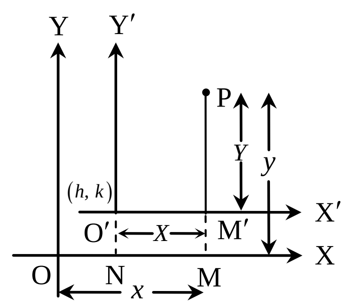
(iv) Replace h by x, and k by y, in the eliminate. The resulting equation would be the equation of the locus of P.

(v) If x and y coordinates of the moving point are obtained in terms of a third variable t (called the parameter), eliminate t to obtain the relation in x and y and simplify this relation. This will give the required equation of locus.

TRANSLATION OF AXES

The translation of axes involves the shifting of the origin to a new point, the new axes remaining parallel to the original axes.

Let OX, OY be the original axes and O' be the new origin. Let coordinates of O' referred to original axes i.e. OX, OY be (h, k).



Let $O'X'$ and $O'Y'$ be drawn parallel to and in the same direction as OX and OY respectively. Let P be any point in the plane having coordinates (x, y) referred to old axes and (X, Y) referred to new axes.

Then, $x = OM = ON + NM = ON + O'M'$ and $y = MP = MM' + M'P = NO' + M'P$

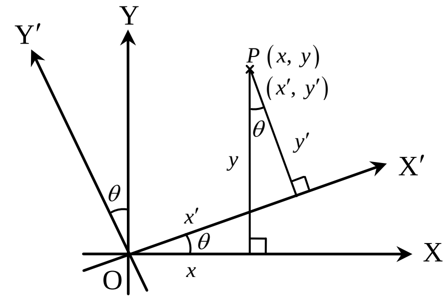
$= h + X = X + h$ $\qquad$ $= k + Y = Y + k.$

Thus, we get $x = X + h,\ y = Y + k \Rightarrow X = x - h,\ Y = y - k.$

Thus, the point whose coordinates were (x, y) has new coordinates $(x - h, y - k)$.

ROTATION OF AXES

ROTATION OF AXES WITHOUT CHANGING THE ORIGIN

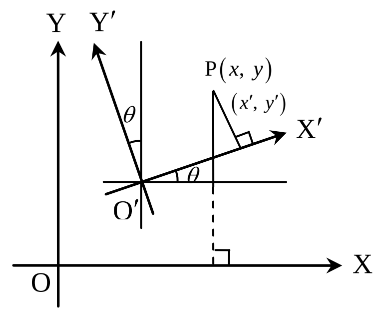


Let OX, OY be the original axes and OX', OY' be the new axes obtained by rotating OX and OY through an angle θ in the anticlockwise sense. Let P be any point in the plane having coordinates (x, y) w.r.t. axes OX and OY and (x', y') w.r.t. axes OX' and OY'.

Then,

$$\begin{cases} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}.$$

CHANGE OF ORIGIN AND ROTATION OF AXES



If origin is changed to $O'(h, k)$ and axes are rotated about the new origin O' by angle θ in the anticlockwise sense such that the new coordinates of $P(x, y)$ become (x', y') then the equations of transformation will be

$$x = h + x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = k + x'\sin\theta + y'\cos\theta.$$

GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form $ax + by + c = 0$, where a, b, c are constants and a, b are not simultaneously zero, always represents a straight line.

SLOPE OF A LINE

If a line makes an angle $\theta\left(\theta \neq \frac{\pi}{2}\right)$ with the positive direction of x-axis, then $\tan\theta$ is the slope or gradient of that line. It is usually denoted by m. *i.e.* $m = \tan\theta$.

Slope of the line joining two points (x_1, y_1) and (x_2, y_2) $= \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

INTERCEPT OF A LINE ON THE AXES

(i) **Intercept of a line on *x*-axis**

If a line cuts x-axis at $(a, 0)$, then a is called the intercept of the line on *x*-axis. $|a|$ is called the length of the intercept of the line on *x*-axis. Intercept of a line on *x*-axis may be positive or negative.

(ii) **Intercept of a line on *y*-axis**

If a line cuts *y*-axis at $(0, b)$, then b is called the intercept of the line on *y*-axis and $|b|$ is called the length of the intercept of the line on *y*-axis. Intercept of a line on *y*-axis may be positive or negative.

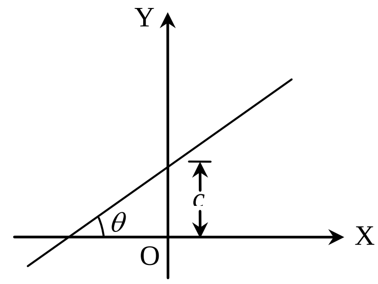
EQUATIONS OF LINES PARALLEL TO AXES

1. **Equation of *x*-axis :** The equation of x-axis is $y = 0$.
2. **Equation of *y*-axis :** The equation of y-axis is $x = 0$.
3. **Equation of a line parallel to *y*-axis :** The equation of the straight line parallel to y-axis at a distance a from it on the positive side of x-axis is $x = a$.

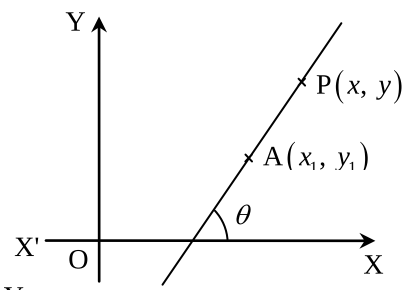
4. **Equation of a line parallel to *x*-axis :** The equation of the straight line parallel to x-axis at a distance b from it on the positive side of y-axis is $y = b.$

EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS

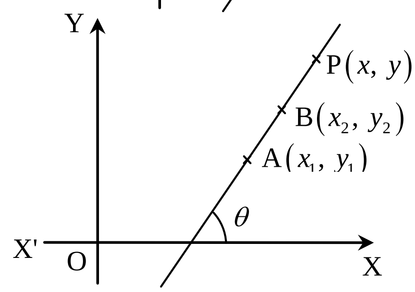
1. **Slope-intercept form**
   The equation of a straight line whose slope is m and which cuts an intercept c on the y-axis is given by $y = mx + c.$



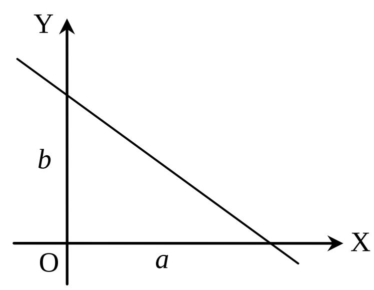
2. **Point-slope form**
   The equation of a straight line passing through the point (x_1, y_1) and having slope m is given by
   $$y - y_1 = m(x - x_1).$$



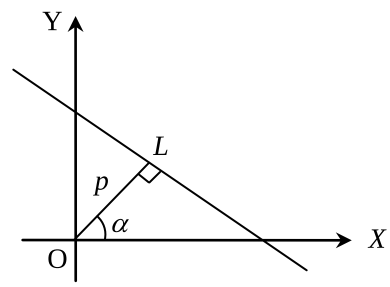
3. **Two-point form**
   The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is given by
   $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$



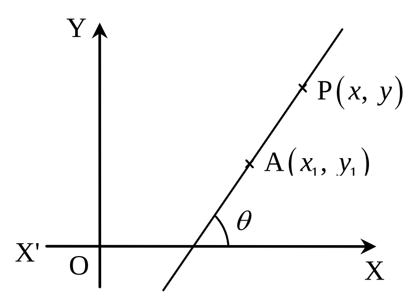
4. **Intercept form**
   The equation of a straight line which cuts off intercepts a and b on x-axis and y-axis respectively is given by $\frac{x}{a} + \frac{y}{b} = 1.$



5. **Normal form (or perpendicular form)**
   The equation of a straight line upon which the length of the perpendicular from the origin is p and the perpendicular makes an angle α with the positive direction of x-axis is given by
   $$x\cos\alpha + y\sin\alpha = p.$$
   Note : In normal form of equation of a straight line p is always taken as positive and α is measured from positive direction of x-axis in anticlockwise direction between 0 and 2π .



6. **Parametric form or symmetric form**
   The equation of a straight line passing through the point (x_1, y_1) and making an angle θ with the positive direction of x-axis is
   $$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \qquad \text{where} \quad 0 \le \theta < \pi$$



where r is the distance of the point $(x,\ y)$ from the point $(x_1,\ y_1)$.

Note : The coordinates of any point on the line at a distance r from the point $A(x_1,\ y_1)$ can be taken as $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$ or $(x_1 - r\cos\theta,\ y_1 - r\sin\theta)$ and $0 \le \theta \le \pi$

REDUCTION OF THE GENERAL EQUATION TO DIFFERENT STANDARD FORMS

1. **Slope-intercept form :** To reduce the equation $Ax + By + C = 0$ to the form $y = mx + c$.

Given equation is $Ax + By + C = 0$ or $y = -\frac{A}{B}x - \frac{C}{B}$

which is of the form $y = mx + c$,

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

Note : Slope of the line $Ax + By + C = 0$ is $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{A}{B}$.

2. **Intercept form :** To reduce the equation $Ax + By + C = 0$ to the form $\frac{x}{a} + \frac{y}{b} = 1$.

This reduction is possible only when $C \ne 0$.
Given equation is $Ax + By + C = 0$

$\Rightarrow\ -\frac{A}{C}x - \frac{B}{C}y = 1,$ where $C \ne 0$ or $\frac{x}{-C/A} + \frac{y}{-C/B} = 1,$ which is of the form $\frac{x}{a} + \frac{y}{b} = 1,$

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$.

3. **Normal form :** To reduce the equation $Ax + By + C = 0$ to the form $x\cos\alpha + y\sin\alpha = p$.

Given equation is $Ax + By + C = 0$ or $Ax + By = -C$ … (1)

CASE 1. When $C < 0,$ i.e. $-C > 0,$ dividing both sides of equation (1) by $\sqrt{A^2 + B^2}$, we get

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}}$$

which is of the form $x\cos\alpha + y\sin\alpha = p$,

where $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$ and $p = -\frac{C}{\sqrt{A^2+B^2}}$

CASE 2. When $C > 0$ i.e. $-C < 0$; from (1) $-Ax - By = C$

or $\frac{-A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$

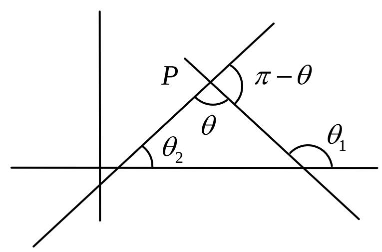
which is of the form $x\cos\alpha + y\sin\alpha = p$,

where $\cos\alpha = -\frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = -\frac{B}{\sqrt{A^2+B^2}}$ and $p = \frac{C}{\sqrt{A^2+B^2}}$.

ANGLE BETWEEN TWO INTERSECTING LINES

The angle θ between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2},$$



provided no line is $\perp$ to x-axis and the acute angle θ is given by $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Note :

(a) If both the lines are $\perp$ to x-axis then the angle between them is $0°$.

(b) If any of the two lines is perpendicular to x-axis, then the slope of that line is infinite.

Let $m_1 = \infty$. Then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$

or $\theta = |90° - \alpha|$ where $\tan\alpha = m_2$

(c) The two lines are parallel if and only if $m_1 = m_2$.

(d) The two lines are $\perp$ if and only if $m_1 \times m_2 = -1$.

CONDITION FOR TWO LINES TO BE COINCIDENT, PARALLEL, PERPENDICULAR OR INTERSECTING

Two lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are

(i) **Coincident,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$; (ii) **Parallel,** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$;

(iii) **Perpendicular,** if $a_1 a_2 + b_1 b_2 = 0$;

(iv) **Intersecting,** if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ i.e. if they are neither co-incident nor parallel.

EQUATION OF A LINE PARALLEL TO A GIVEN LINE

The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + k = 0$, where k is a constant.

Thus to write the equation of any line parallel to a given line, do not change the coefficient of x and y and change the constant term only.

EQUATION OF A LINE PERPENDICULAR TO A GIVEN LINE

The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + k = 0$, where k is a constant. Thus to write the equation of any line perpendicular to a given line interchange the coefficients of x and y then change the sign of any one of them and finally change the constant term.

POINT OF INTERSECTION OF TWO GIVEN LINES

Let the two given lines be $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$.

Solving these two equations, the point of intersection of the given two lines is given by

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right).$$

INTERIOR ANGLES OF A TRIANGLE : To find the interior angles of a triangle arrange the slopes of the sides in decreasing order i.e., $m_1 > m_2 > m_3$. Then apply

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}, \quad \tan\beta = \frac{m_2 - m_3}{1 + m_2 m_3}, \quad \tan\gamma = \frac{m_3 - m_1}{1 + m_3 m_1}$$

LINES THROUGH THE INTERSECTION OF TWO GIVEN LINES

The equation of any line passing through the point of intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is $(a_1 x + b_1 y + c_1) + k(a_2 x + b_2 y + c_2) = 0$, where k is a parameter. The value of k can be obtained by using one more conditions which the required line satisfies.

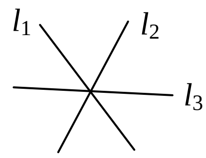
CONDITIONS OF CONCURRENCE

The family of given lines are said to be concurrent if they meet in a point.

WORKING RULE TO PROVE THAT THREE GIVEN LINES ARE CONCURRENT

1. The three lines $a_1x+b_1y+c_1=0,\quad a_2x+b_2y+c_2=0,\quad a_3x+b_3y+c_3=0$

   are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$



2. The three lines $P=0,\ Q=0$ and $R=0$ are concurrent if there exist constants $l,\ m$ and $n,$ not all zero at the same time, such that $lP+mQ+nR=0.$

   This method is particularly useful in theoretical results.

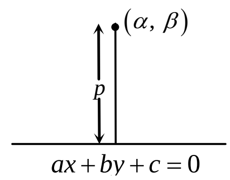
POSITION OF TWO POINTS RELATIVE TO A LINE

Two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$ are on the same side or on opposite sides of the line $ax+by+c=0$ according as the expressions ax_1+by_1+c and ax_2+by_2+c have same sign or opposite signs.

LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

The length of the perpendicular from the point $(\alpha,\ \beta)$ to the line $ax+by+c=0$ is given by

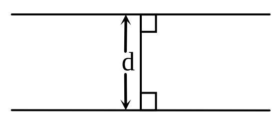
$$p=\frac{|a\alpha+b\beta+c|}{\sqrt{a^2+b^2}}.$$



DISTANCE BETWEEN TWO PARALLEL LINES

The distance between two parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is given by

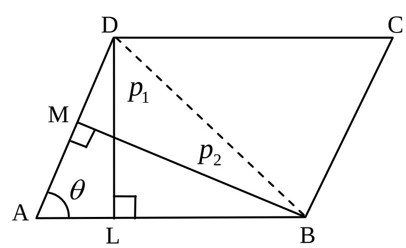
$$d=\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}.$$



Note : (a) The distance between two parallel lines can also be obtained by taking a suitable point (take $y=0$ and find x or take $x=0$ and find y) on one straight line and then finding the length of the perpendicular from this point to the second line.

(b) Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$Area=\frac{p_1p_2}{\sin\theta}=p_1p_2\operatorname{cosec}\theta$$



where $p_1=DL=$ distance between lines AB and $CD,$

$p_2=BM=$ distance between lines AD and $BC,$

$\theta=$ angle between adjacent sides AB and $AD.$

In the case of a rhombus, $p_1=p_2$

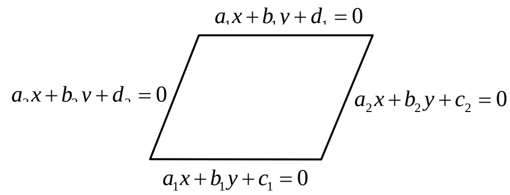
$\therefore$ Area of rhombus $=\dfrac{p_1^2}{\sin\theta}.$

Also, area of rhombus $= \frac{1}{2} d_1 d_2$

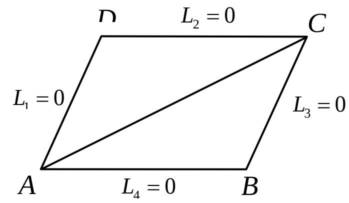
where d_1 and d_2 are the lengths of two $\perp$ diagonals of a rhombus.

(c) If the equation of sides of a parallelogram are as shown then its area is given by

$$\left| \frac{(c_1 - d_1)(c_2 - d_2)}{a_1 b_2 - a_2 b_1} \right|$$



(d) If equation of sides of a parallelogram are $L_1 = 0,\ L_2 = 0,\ L_3 = 0,\ L_4 = 0$ then $L_2L_3 - L_1L_4 = 0$ represents the diagonal BD. Again $L_1L_2 - L_3L_4 = 0$ represents the diagonal AC.



EQUATIONS OF STAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

The equations of the straight lines which pass through a given point $(x_1,\ y_1)$ and make a given angle α with the given straight line $y = mx + c$ are $y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$

EQUATIONS OF THE BISECTORS OF THE ANGLES BETWEEN THE LINES

The equations of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

TO FIND THE EQUATION OF THE BISECTOR OF THE ACUTE AND OBTUSE ANGLE BETWEEN TWO LINES

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ … (1)

and $a_2x + b_2y + c_2 = 0$ … (2)

where $c_1 > 0$ and $c_2 > 0$.

Then the equation $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

is the equation of bisector containing origin.

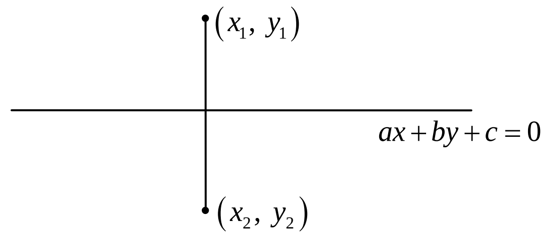
Similarly, the equation $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = - \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

is the equation of bisector not containing origin.

Note : If $a_1a_2 + b_1b_2 > 0,$ then the origin lies in obtuse angle i.e., the bisector containing origin is obtuse angle bisector and if $a_1a_2 + b_1b_2 < 0,$ then the origin lies in acute angle i.e., the bisector containing origin is acute angle bisector.

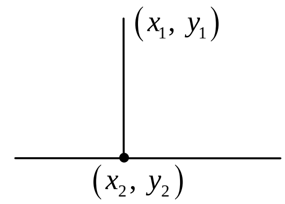
1. **IMAGE OF A POINT :** The co-ordinate of image $(x_2,\ y_2)$ of a given point $(x_1,\ y_1)$ in the line $ax+by+c=0$ are given by

$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$



2. **FOOT OF THE PERPENDICULAR :** The co-ordinate of foot of perpendicular $(x_2,\ y_2)$ of a given point $(x_1,\ y_1)$ on the line $ax+by+c=0$ are given by

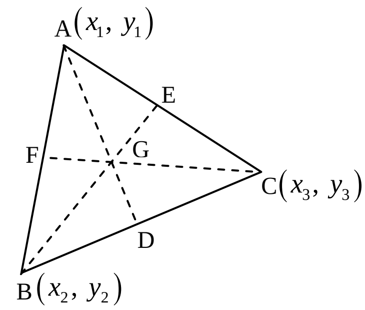
$$\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$$



STANDARD POINTS OF A TRIANGLE

Centroid of a Triangle

The point of intersection of the medians of the triangle is called the centroid of the triangle. The centroid divides the medians in the ratio $2:1$.



The coordinates of the centroid of a triangle with vertices $(x_1,\ y_1),\ (x_2,\ y_2)$ and $(x_3,\ y_3)$ are

$$\left(\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}\right).$$

(i) If P is any point in the plane of the triangle ABC and G is the centroid then

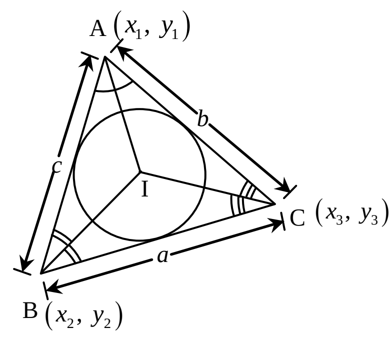
$$PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2$$

(ii) If G is the centroid of the triangle ABC, then

$$AB^2+BC^2+CA^2=3\left[GA^2+GB^2+GC^2\right]$$

Incentre of a Triangle

The point of intersection of the internal bisectors of the angles of a triangle is called the *incentre of the triangle*.



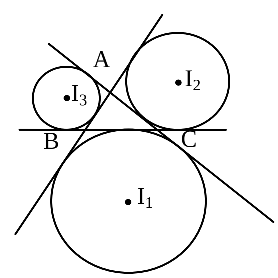
The coordinates of the incetre of a triangle with vertices $(x_1,\ y_1),\ (x_2,\ y_2)$ and $(x_3,\ y_3)$ are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\ \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

Ex-centres of a Triangle

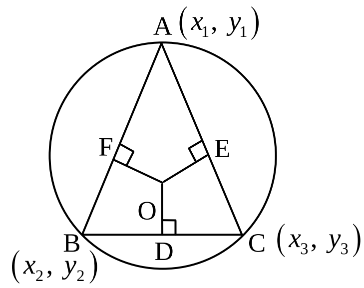
A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.

Let ABC be a triangle then there are three excircles, with three excentres $I_1,\ I_2,\ I_3$ opposite to vertices A, B and C respectively. If the vertices of triangle are $A(x_1,\ y_1),\ B(x_2,\ y_2)$ and $C(x_3,\ y_3)$ then



$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$
$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$
$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right).$$

CIRCUMCENTRE



The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of the sides of a triangle. It is the centre of the circle which passes through the vertices of the triangle and so its distance from the vertices of the triangle is same and this distance is known as the circum-radius of the triangle.

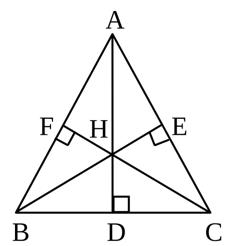
If angles of triangle ΔABC i.e. A, B, C and vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then circumcentre of the triangle ABC is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

ORTHOCENTRE

The orthocentre of a triangle is the point of intersection of altitudes.

If angles of a ΔABC, i.e. A, B and C and vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given, then orthocentre of ΔABC is



$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

Note :

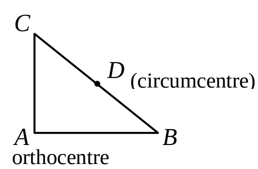
(i) If H is the orthocentre of ΔABC then

orthocentre of ΔAHB is C

orthocentre of ΔBHC is A

orthocentre of ΔAHC is B

(ii) If any two lines out of three lines i.e. AB, BC and CA are perpendicular, then orthocentre is the point of intersection of two perpendicular lines i.e., in right angled Δ the right angled vertex is the orthocentre of Δ and mid-point of hypotneuse is circumcentre



Pair of Straight line

Chapter 25

Pair of Straight Line

1. **SECOND DEGREE HOMOGENEOUS EQUATION**

A second degree homogenous equation is $ax^2+2hxy+by^2=0....(1)$ will represent pair of straight lines passing through origin if $h^2 \ge ab.$ In case of equality, lines will be identical.

If $h^2<ab,$ then (1) will represent the point 'origin' only. Slopes of the lines given by (1) are given by $m_1, m_2 = \dfrac{-h \pm \sqrt{h^2-ab}}{b}$; $m_1+m_2=-\dfrac{2h}{b}$ and $m_1m_2=\dfrac{a}{b}$

lines given by (1) are $y=m_1x$ and $y=m_2x.$ Angle between the lines given by (1) is given by $\tan\theta=\left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|.$ If $a+b=0$, then lines given by (1) are perpendicular to each other.

Equations of the angle bisectors of the angles formed by (1) are given by $\dfrac{x^2-y^2}{a-b}=\dfrac{xy}{h}.$

2. **IDENTIFICATION OF CURVES**

General equation of second degree is $ax^2+2hxy+by^2+2gx+2fy+c=0$ ....(1)

and $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2.$

Case I : (i) If $\Delta=0$ and $h^2>ab,$ then (1) represent intersecting lines.

(ii) If $\Delta=0$ and $h^2=ab,$ then (1) represents pair of parallel straight lines or coincident lines.

(iii) If $\Delta=0$ and $h^2<ab,$ then (1) represent a point only which is given by $\left(\dfrac{hf-bg}{ab-h^2},\dfrac{hg-af}{ab-h^2}\right).$

Case II : If $\Delta \ne 0,$ then

(i) $a=b$ and $h=0$, then (1) is circle

(ii) $h^2=ab$, then (1) is parabola

(iii) $h^2<ab,$ then (1) is ellipse

(iv) $h^2>ab,$ then (1) is hyperbola.

3. **INFORMATION ABOUT PAIR OF STRAIGHT LINES**

General equation of second degree is

$ax^2+2hxy+by^2+2gx+2fy+c=0$ if $abc+2fgh-af^2-bg^2-ch^2=0$

and $h^2>ab,$ then (1) represents pair of intersecting lines. Formula for the slopes of the lines, sum and product of the slopes, angles between the lines and condition for the perpendicularity of lines will remain same as it was in case of second degree homogeneous equation.

Let point of intersection of lines given by (1) be $(x_1,y_1),$ then $x_1=\dfrac{hf-bg}{ab-h^2}$ and $y_1=\dfrac{hg-af}{ab-h^2}.$

4. **Equation of the angle bisectors** of the angles formed by (1) are given by

$$\frac{(x-x_1)^2-(y-y_1)^2}{a-b}=\frac{(x-x_1)(y-y_1)}{h}.$$

5. **DISTANCE BETWEEN PAIR OF PARALLEL LINES**

If $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ *and* $h^2 = ab,$ then (1) represents pair of parallel lines.

Distance between these pair of parallel lines is $= 2\sqrt{\dfrac{g^2 - ac}{a(a+b)}} = 2\sqrt{\dfrac{f^2 - bc}{b(a+b)}}$

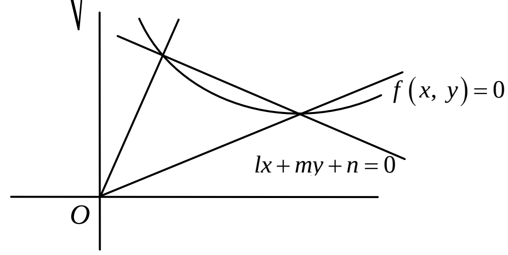
6. **Join equation of** pair of straight lines passing through origin and point of intersection of

$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$

and $lx + my + n = 0 \quad \ldots (2)$

are given by

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(-\frac{lx+my}{n}\right) + c\left(-\frac{lx+my}{n}\right)^2 = 0.$$



Circle

Chapter 26

CIRCLE

A circle is the locus of point which moves in a plane such that its distance from a fixed point is constant. The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

STANDARD EQUATION OF A CIRCLE

1. The equation of a circle with the centre at (α, β) and radius a, is $(x-\alpha)^2+(y-\beta)^2=a^2$
2. If the centre of the circle is at the origin and the radius is a, then the equation of circle is $x^2+y^2=a^2$.

GENERAL EQUATION OF A CIRCLE

The general equation of a circle is of the form $x^2+y^2+2gx+2fy+c=0$, … (1)

Where g, f and c are constants.

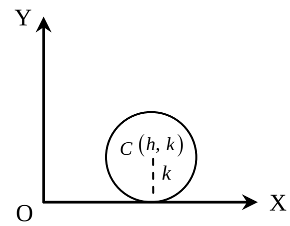
The coordinates of the centre are $(-g, -f)$ and radius $=\sqrt{g^2+f^2-c}$.

CONDITIONS FOR GENERAL EQUATION OF SECOND DEGREE TO REPRESENT A CIRCLE

A general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ in x, y represents a circle if

1. Coefficient of $x^2 =$ coefficient of y^2 i.e. $a=b$,
2. Coefficient of xy is zero i.e. $h=0$.

DIFFERENT FORMS OF THE EQUATION OF A CIRCLE

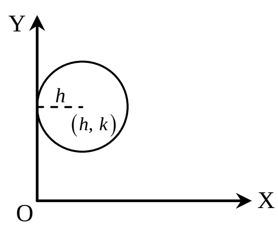


1. **Circle with centre at the point *(h, k)* and which touches the axis of *x***

   Since the circle touches the x-axis, the radius of the circle $=k$.

   $\therefore$ Equation of the circle is $(x-h)^2+(y-k)^2=k^2$

   or $x^2+y^2-2hx-2ky+h^2=0$



2. **Circle with centre at the point *(h, k)* and which touches the axis of *y***

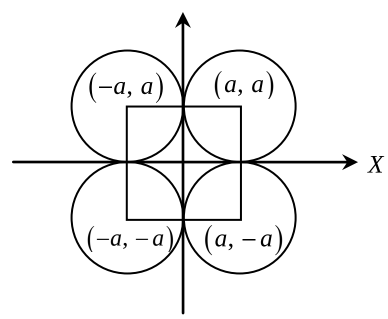
   Since the circle touches the y-axis, the radius of the circle $=h$

   $\therefore$ Equation of the circle is $(x-h)^2+(y-k)^2=h^2$

   or $x^2+y^2-2hx-2ky+k^2=0$.

3. **Circle with radius *a* and which touches both the coordinate axes**

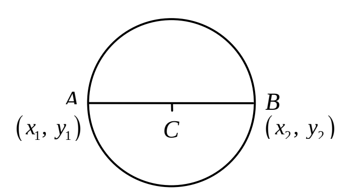
   Since the centre of the circle may be in any of the four quadrants, therefore it will be any one of the four points $(\pm a, \pm a)$. Thus, there are four circles of radius a touching both the coordinate axes, and their equations are $(x\pm a)^2+(y\pm a)^2=a^2$ or $x^2+y^2\pm 2ax\pm 2ay+a^2=0$.



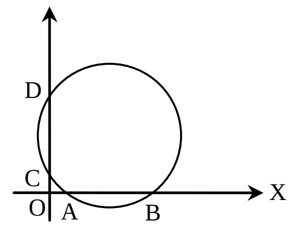
CIRCLE ON A GIVEN DIAMETER

The equation of the circle drawn on the line segment joining two given points (x_1, y_1) and (x_2, y_2) as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

Its centre is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and radius is $\frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$



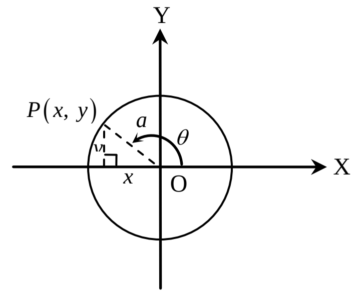
INTERCEPTS MADE BY A CIRCLE ON THE AXES



1. The length of the intercept made by the circle $x^2+y^2+2gx+2fy+c=0$ on

   (a) x-axis $= AB = 2\sqrt{g^2-c}$ (b) y-axis $= CD = 2\sqrt{f^2-c}$
2. Intercepts are always positive.
3. If the circle touches x-axis then $AB=0$ $\therefore\ c=g^2$.
4. If the circle touches y-axis, then $CD=0$ $\therefore\ c=f^2$.
5. If the circle touches both the axes, then $AB=0=CD$ $\therefore\ c=g^2=f^2$.

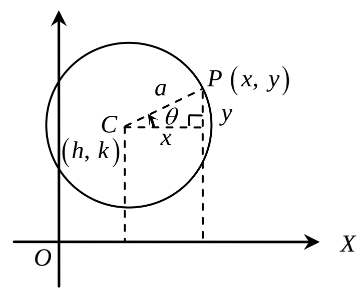
PARAMETRIC EQUATIONS OF A CIRCLE



1. The parametric equations of a circle $x^2+y^2=a^2$ are $x=a\cos\theta$, $y=a\sin\theta$, $0\le\theta<2\pi$.

   θ is called parameter and the point $P(a\cos\theta,\ a\sin\theta)$ is called the point 'θ' on the circle $x^2+y^2=a^2$.

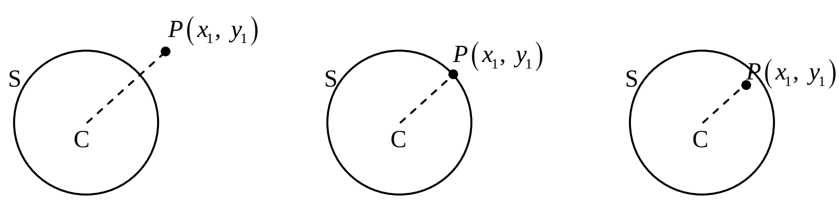
   Thus, the coordinates of any point on the circle $x^2+y^2=a^2$ may be taken as $(a\cos\theta,\ a\sin\theta)$.



2. The parametric equations of a circle $(x-h)^2+(y-k)^2=a^2$ are $x=h+a\cos\theta$, $y=k+a\sin\theta$, $0\le\theta<2\pi$ is called the point 'θ' on this circle. Thus the coordinates of any point on this circle may be taken as $(h+a\cos\theta,\ k+a\sin\theta)$.

POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let $S\equiv x^2+y^2+2gx+2fy+c=0$, be a circle and $P(x_1, y_1)$ be a point in the plane of S, then $S_1\equiv x_1^2+y_1^2+2gx_1+2fy_1+c$. The point $P(x_1, y_1)$ lies outside, on or inside the circle S according as $S_1 >, =$ or <0.



Note : Let S be a circle and $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane of S, then they lie

(a) on the same side of S iff S_1 and S_2 have same sign,

(b) on the opposite sides of S iff S_1 and S_2 have opposite signs.

CIRCLE THROUGH THREE POINTS

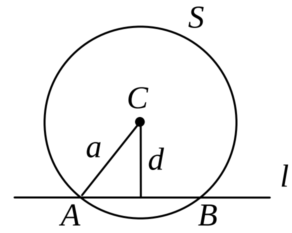
The equation of the circle through three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

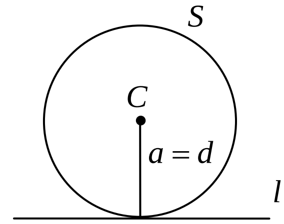
INTERSECTION OF A LINE AND A CIRCLE

Let S be a circle with centre C and radius a. Let l be any line in the plane of the circle and d be the perpendicular distance from C to the line l, then

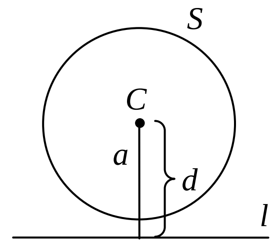
(a) l intersects S in two distinct points iff $d < a$.



(b) l intersects S in one and only one point iff $d = a$ i.e. the line touches the circle iff perpendicular distance from the centre to the line is equal to the radius of the circle.

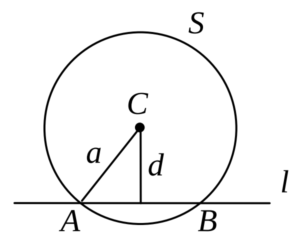


(c) l does not intersect S iff $d > a$.



LENGTH OF THE INTERCEPT MADE BY A CIRCLE ON A LINE

If the line l meets the circle S with centre C and radius 'a' in two distinct points A and B and if d is the perpendicular distance of C from the line l, the length of the intercept made by the circle on the line $= |AB| = 2\sqrt{a^2 - d^2}$.



To find the point of intersection of a line $y = mx + c$ with a circle $x^2 + y^2 = a^2$ we need to solve both the curves i.e. roots of equation $x^2 + (mx + c)^2 = a^2$ gives x coordinates of the point of intersection. Now following cases arise :

(i) Discriminant $> 0 \Rightarrow$ two distinct and real points of intersection.

(ii) Discriminant $= 0 \Rightarrow$ coincident roots i.e. line is tangent to the circle.

(iii) Discriminant $< 0 \Rightarrow$ no real point of intersection.

TANGENT TO A CIRCLE AT A GIVEN POINT

1. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) on it is

$$xx_1 + yy_1 = a^2.$$

2. Equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on it is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

3. Equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ on it is

$$x\cos\theta + y\sin\theta = a \qquad \text{[Parametric form of equation of tangent]}$$

Note : The equation of the tangent at the point (x_1, y_1) on the circle $S = 0$ is $T = 0$.

EQUATION OF THE TANGENT IN SLOPE FORM

The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1 + m^2}$.

The coordinates of the point of contact are $\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$.

CONDITION OF TANGENCY

The straight line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1+m^2}$.

Note : A line will touch a circle if and only if the length of the $\perp$ from the centre of the circle to the line is equal to the radius of the circle.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal to a circle, at any point on the circle, is a straight line which is $\perp$ to the tangent to the circle at that point and always passes through the centre of the circle.

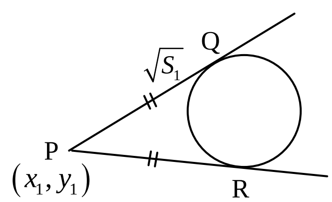
1. Equation of the normal to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) on it is

$$\frac{x}{x_1} = \frac{y}{y_1}.$$

2. Equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on it is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$

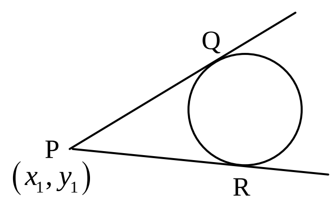
LENGTH OF TANGENTS



Let PQ and PR be two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then PQ = PR and the length of tangent drawn from point P is given by

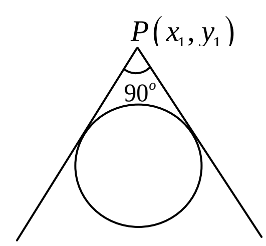
$$\text{PQ} = \text{PR} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}.$$

PAIR OF TANGENTS



Form a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ Their combined equation is $SS_1 = T^2$ where S= 0 is the equation of circle, T=0 is the equation of tangents at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

DIRECTOR CIRCLE



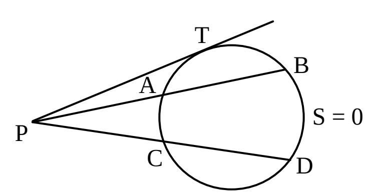
The locus of the point of intersection of two perpendicular tangents to a circle is called the Director circle.

Let the circle be $x^2 + y^2 = a^2$. then equation of director circle is $x^2 + y^2 = 2a^2$.Obviously director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

Director circle of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$$

POWER OF A POINT WITH RESPECT TO A CIRCLE



Let $P(x_1, y_1)$ be point and secant PAB, drawn.

The power of $P(x_1, y_1)$ w.r.t.

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

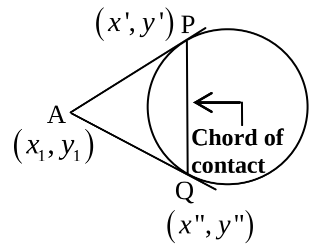
is equal to $PA.PB$, which is $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore$ Power remains constant for the circle i.e. independent of A and B

$\therefore$ $PA.PB = PC.PD = PT^2 =$ square of the length of a tangent.

[Power of a point P is positive, negative or zero according to position of P outside, inside or on the circle respectively]

CHORD OF CONTACT OF TANGENTS



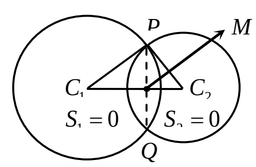
1. **Chord of contact:** The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents

2. **Equation of chord of contact:**
The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$
Equation of chord of contact at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. It is clear from above that the equation to the chord of contact coincides with the equation of the tangent, if point (x_1, y_1) lies on the circle.
The length of chord of contact = $2\sqrt{r^2 - p^2}$; (p being length of perpendicular from centre of the chord)
Area of ΔAPQ *is given by* $\dfrac{a\left(x_1^2 + y_1^2 - a^2\right)^{3/2}}{x_1^2 + y_1^2}$

3. **Equation of the chord bisected at a given point:**
The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$ *. i.e.,* $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

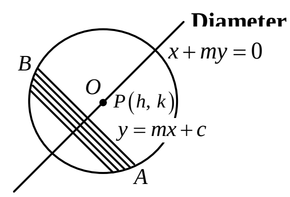
COMMON CHORD OF TWO CIRCLES

1. **Definition :** The chord joining the points of intersection of two given circles is called their common chord.
2. **Equation of common chord :** The equation of the common chord of two circles
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ … (i)
and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ … (ii)
is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$ i.e., $S_1 - S_2 = 0.$



3. **Length of the common chord :** $PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$
Where $C_1P =$ radius of the circle $S_1 = 0$ and $C_1M =$ length of the perpendicular from the centre C_1 to the common chord PQ.

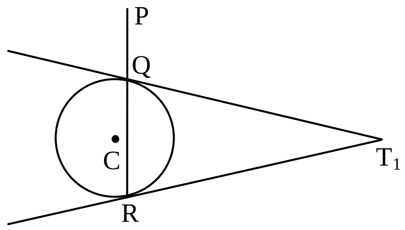
DIAMETER OF A CIRCLE

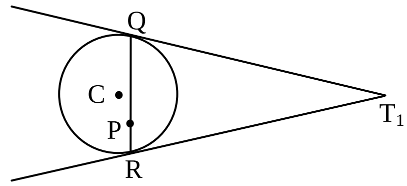


The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.
The equation of the diameter bisecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is $x + my = 0.$

POLE AND POLAR

If from a point P any straight line is drawn to meet the circle in Q and R and if tangents to the circle at Q and R meet in T_1, then the locus of T_1 is called the polar of P with respect to the circle.





The point P is called the pole of its polar.

The polar of the point $P(x_1, y_1)$ w.r.t. the circle $S=0$ is given by $T=0$.

i.e. $xx_1 + yy_1 + = a^2$ for the circle $x^2+y^2=a^2$ and

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$ for the circle $x^2+y^2+2gx+2fy+c=0$.

Note :

(a) If the point P lies outside the circle, then the polar and the chord of contact of this point P are same straight line.

(b) If the point P lies on the circle, then the polar and the tangent to the circle at P are same straight line.

(c) The coordinates of the pole of the line $lx+my+n=0$ with respect to the circle $x^2+y^2=a^2$ are $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.

CONJUGATE POINTS

Two points are said to be conjugate points with respect to a circle if the polar of either passes through the other

CONJUGATE LINES

Two straight lines are said to be conjugate lines if the pole of either lies on the other.

Common tangents to two circles

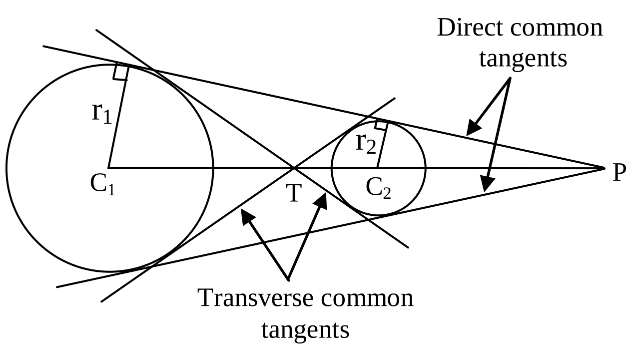
Different cases of intersection of two circle :

Let the two circles be $(x-x_1)^2 + (y-y_1)^2 = r_1^2$ … (i)

and $(x-x_2)^2 + (y-y_2)^2 = r_2^2$ … (ii)

With centres $C_1(x_1, y_1)$ *and* $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

Case I : When $C_1C_2 > r_1 + r_2$ *i.e.,* the distance between the centres is greater than the sum of radii.



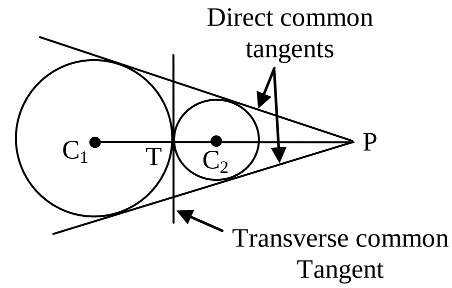
In this case four common tangents can be drawn to the two circles, in which two are direct common tangents and the other two are transverse common tangents.

The points P, T of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

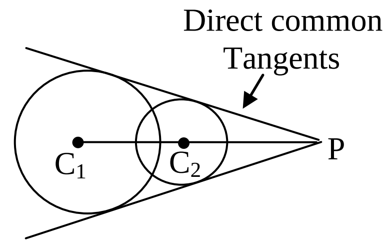
$\dfrac{C_1P}{C_2P} = \dfrac{r_1}{r_2}$ (externally) and $\dfrac{C_1T}{C_2T} = \dfrac{r_1}{r_2}$ (internally) Hence, the ordinates of P and T are

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2} \right) \text{ and } T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \right).$$

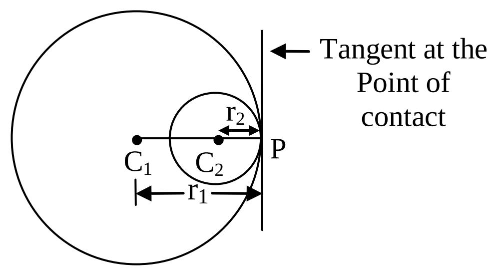
Case II : When $C_1C_2 = r_1 + r_2$ *i.e.,* the distance between the centres is equal to the sum of radii.
In this case two direct common tangents are real and distinct while transverse tangents are coincident.



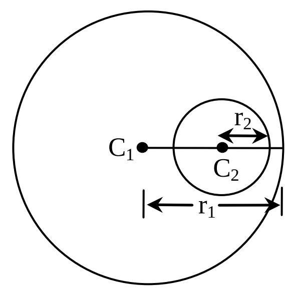
Case III : When $C_1C_2 < r_1 + r_2$ *i.e.,* the distance between the centres is less than sum of radii.
In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.



Case IV : When $C_1C_2 = |r_1 - r_2|$, *i.e.,* the distance between the centres is equal to the difference of the radii.
In this case two tangents are real and coincident while the other two tangents are imaginary.



Case V : When $C_1C_2 < |r_1 - r_2|$, *i.e.,* the distance between the centres is less than the difference of the radii.
In this case, all the four common tangents are imaginary.



WORKING RULE TO FIND DIRECT COMMON TANGENTS

Step 1 : Find the coordinates of centres C_1, C_2 and radii r_1, r_2 of the two given circles.

Step 2 : Find the coordinates of the point, say P dividing C_1C_2 externally in the ratio $r_1 : r_2$. Let $P \equiv (h, k)$.

Step 3 : Write the equation of any line through $P(h, k)$ i.e. $y - k = m(x - h)$ … (1)

Step 4 : Find the two values of m, using the fact that the length of the perpendicular on (1) from the centre C_1 of one circle is equal to its radius r_1.

Step 5 : Substituting these values of 'm' in (1), the equation of the two direct common tangents can be obtained.

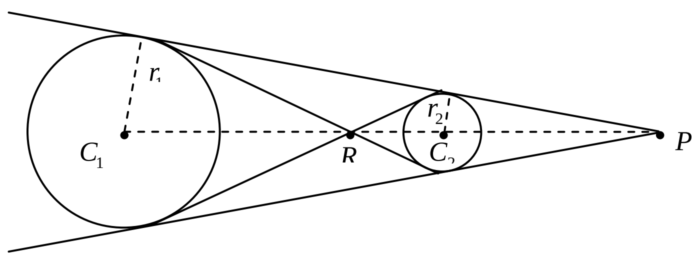
Note :

(a) The direct common tangents to two circles meet on the line joining centres C_1 and C_2, and divide it externally in the ratio of the radii.

(b) The transverse common tangents also meet on the line of centres C_1 and C_2, and divide it internally in the ratio of the radii.

WORKING RULE TO FIND TRANSVERSE COMMON TANGENTS

All the steps except the 2nd step are the same as above. Here in the second step the point $R(h, k)$ will divide C_1C_2 internally in the ratio $r_1 : r_2$.



Note :

(a) When two circles are real and non-intersecting, 4 common tangents can be drawn.

(b) When two circles touch each other externally, 3 common tangents can be drawn to the circles.

(c) When two circles intersect each other at two real and distinct points, two common tangents can be drawn to the circles.

(d) When two circle touch each other internally one common tangent can be drawn to the circles.

IMAGE OF THE CIRCLE BY THE LINE MIRROR

Let the circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and the line be $L = lx + my + n = 0$.

The radius of the image circle will be the same as that of the given circle.

Let the centre of the image circle be (x_1, y_1).

Slope of C_1C_2 $\times$ slope of line $L = -1$ … (1)

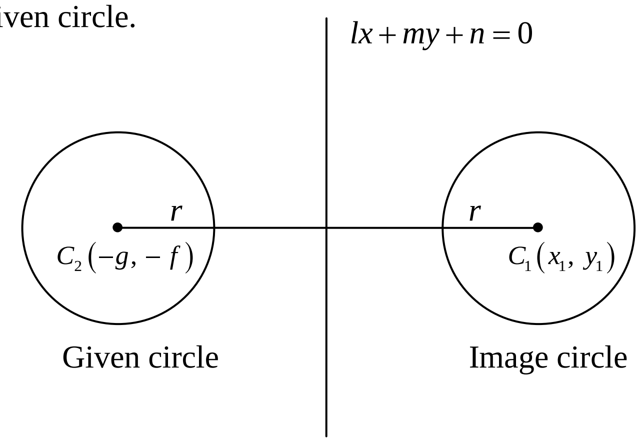
and midpoint of C_1C_2 lies on $lx + my + n = 0$

$$\Rightarrow \quad l\left(\frac{x_1 - g}{2}\right) + m\left(\frac{y_1 - f}{2}\right) + n = 0 \qquad \ldots (2)$$

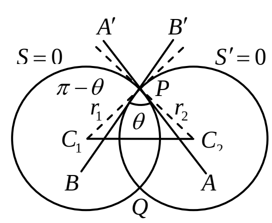
Solving (1) and (2), we get (x_1, y_1).

$\Rightarrow$ Required image circle will be

$$(x - x_1)^2 + (y - y_1)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2.$$



ANGLE OF INTERSECTION OF TWO CIRCLES



The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection.

If $S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

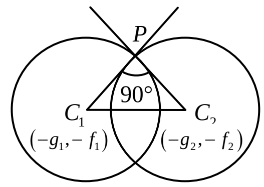
$S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by

$$\cos\theta = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right|$$

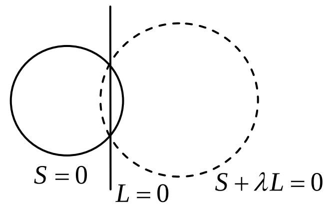
or $$\cos\theta = \left|\frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\sqrt{g_2^2 + f_2^2 - c_2}}\right|.$$

Condition of Orthogonality : If the angle of intersection of the two circles is a right angle $(\theta = 90°)$, then such circles are called orthogonal circles and condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

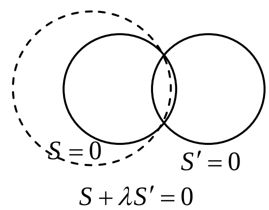


FAMILY OF CIRCLES

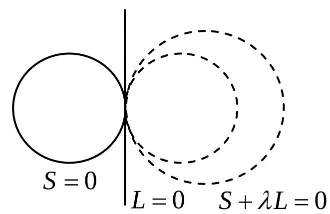
1. The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$, (where λ is a parameter)



2. The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$). But it is better to find first the equation of common $S - S' = 0$ and then use $S + \lambda(S - S') = 0$

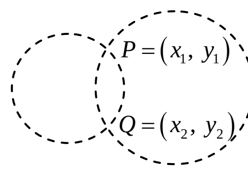


3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)

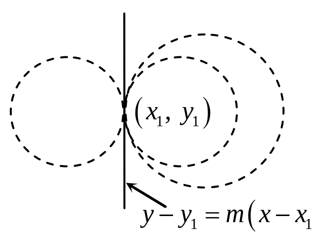


4. The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \quad \text{(where } \lambda \text{ is a parameter)}$$

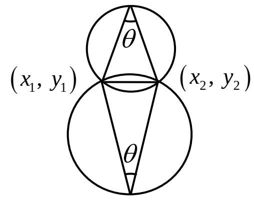


5. The equation of family of circles, which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is $(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$
And if m is infinite, the family of circles is
$(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$, (where λ is a parameter)



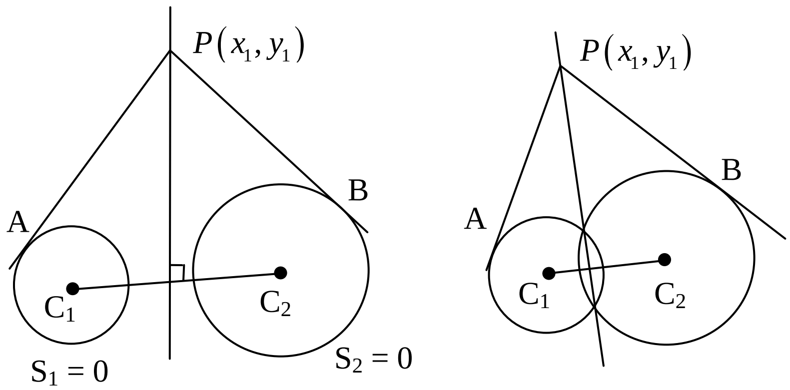
6. Equation of the circles given in diagram is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)\pm\cot\theta\{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)\}=0$$



RADICAL AXIS

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.



The equation of the radical axis of the two circle is $S_1 - S_2 = 0$ i.e.,

$$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0, \qquad \text{which is a straight line.}$$

PROPERTIES OF RADICAL AXIS

1. The radical axis and common chord are identical for two intersecting circles.
2. The radical axis of two circles is perpendicular to the line joining their centres.
3. Radical centre : The radical axis of three circles taken in pairs meet at a point, called the radical centre of the circles. Coordinates of radical centre can be found by solving the equations

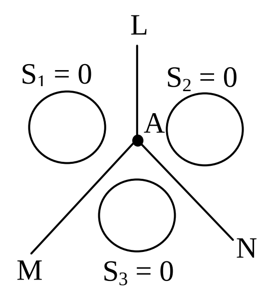
$$S_1 = S_2 = S_3 = 0.$$

4. The radical centre of three circle described on the sides of a triangle as diameters is the orthocentre of the triangle.
5. If two circles cut a third circle orthogonally, then the radical axis of the two circles pass through the centre of the third circle.
6. The radical axis of the two circles will bisect their common tangents.

RADICAL CENTRE

The radical axes of three circles, taken in pairs, meet in a point, which is called their radical centre. Let the three circles be

$$S_1 = 0 \ \ldots\text{(i)}, \quad S_2 = 0 \ \ldots\text{(ii)} \ \text{ and } \ S_3 = 0 \ \ldots\text{(iii)}$$



Let the straight lines i.e., AL and AM meet in A.

The equation of any straight line passing through A is

$$(S_1-S_2)+\lambda(S_3-S_1)=0, \text{ where } \lambda \text{ is any constant.}$$

For $\lambda = 1$, this equation becomes $S_2 - S_3 = 0$, which is, equation of AN.

Thus the third radical axis also passes through the point where the straight lines AL and AM meet.

In the above figure A is the radical centre.

PROPERTIES OF RADICAL CENTRE

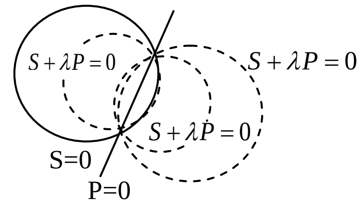
Co-axial system of circles

A system (or a family) of circles, every pair of which have the same radical axis, are called co-axial circles.

1. The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are

$$P \equiv lx + my + n = 0, S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).

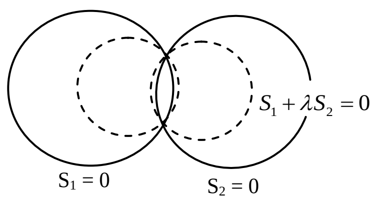


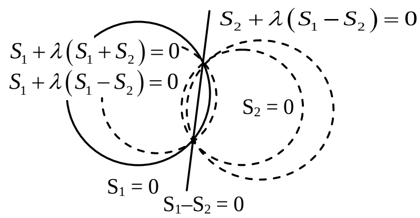
2. The equation of a co-axial system of circles, where the equation of any two circles of the system are

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \text{ and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

respectively, is $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda_1(S_1 - S_2) = 0$

Other form $S_1 + \lambda S_2 = 0, \ (\lambda \neq -1)$





3. The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant.

LIMITING POINTS

Limiting points of a system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called point circles).

Let the circle is $x^2 + y^2 + 2gx + c = 0$ … (i)

where g is a variable and c is a constant.

$\therefore$ Centre and the radius of (i) are $(-g, 0)$ and $\sqrt{(g^2 - c)}$ respectively. Let $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$.

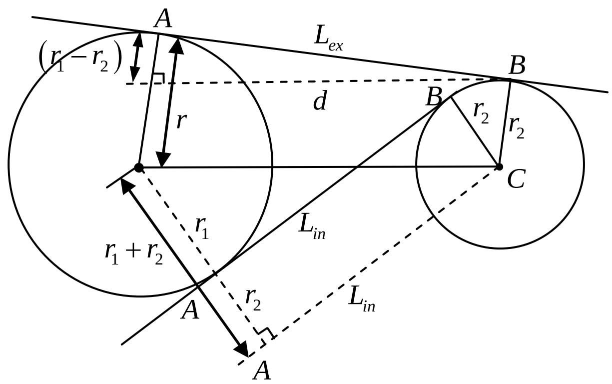
TIPS & TRICKS

Length of an external common tangent and internal common tangent to two circles is given by length of external common tangent

$$L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2}$$

and length of internal common tangent $L_{in} = \sqrt{d^2 - (r_1 + r_2)^2}$ [Applicable only when $d > (r_1 + r_2)$]

where d is the distance between the centres of two circles i.e., $|C_1C_2| = d$ and r_1 and r_2 are the radii of two circles.

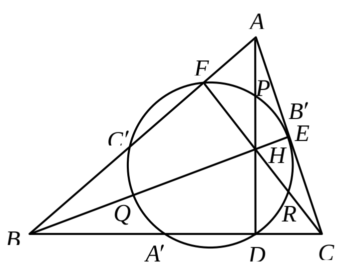


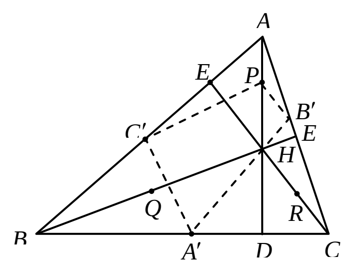
Nine-point circle : The circle through the midpoints of the sides of a triangle passes through the feet of the altitudes and the midpoints of the lines joining the orthocentre to the vertices. This circle is called the nine-point circle of the triangle.

Note :

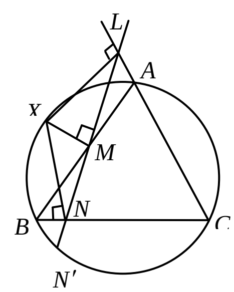
(a) The radius of the nine point circle is half the radius of the circumcircle of the triangle ABC

(b) Centre is midpoint of the line segment joining orthocenter and circumcentre.

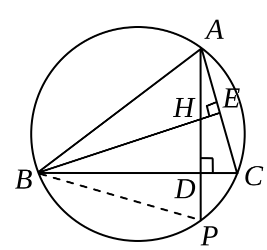




Simson's line : The feet L, M, N of the perpendicular on the sides BC, CA, AB of any ΔABC from any point X on the circumcircle of the triangle are collinear. The line LMN is called the ***Simson's line*** or the ***pedal line*** of the point X with respect to ΔABC.



If H is the orthocentre of ΔABC and AH produced meets BC at D and the circumcircle of ΔABC at P, then $HD = DP$.

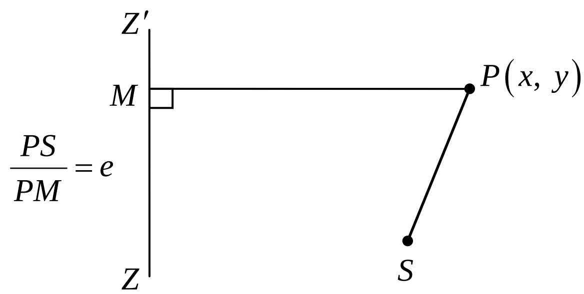


Conic Section

Chapter 27

CONIC SECTION

A conic section or the conic is the locus of a point which moves in a plane is such a way that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line.



The fixed point is called the **focus** and the fixed line is called the **directrix** of the conic. The constant ratio is called the **eccentricity** of the conic and is denoted by *e*.

If $e=1$, the conic is called **Parabola**.
If $e<1$, the conic is called **Ellipse**.
If $e>1$, the conic is called **Hyperbola**.
If $e=0$, the conic is called **Circle**.
If $e\to\infty$, the conic is called **pair of straight lines**.

IMPORTANT TERMS

AXIS

The straight line passing through the focus and perpendicular to the directrix of the conic is known as its axis.

VERTEX

A point of intersection of a conic with its axis is known as a vertex of the conic.

CENTRE

The point which bisects every chord of the conic passing through it, is called the centre of the conic.

FOCAL CHORD

A chord passing through the focus is known as focal chord of the conic.

LATUS RECTUM

The focal chord which is perpendicular to the axis is known as latus rectum of the conic.

DOUBLE ORDINATE

A chord of the conic which is perpendicular to the axis is called the double ordinate of the conic.

GENERAL EQUATION

The general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents

1. a pair of straight lines if $\Delta=0$ where $\Delta=abc+2fgh-af^2-bg^2-ch^2$

   or $\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$,

2. a circle if $\Delta\neq 0$, $a=b$ and $h=0$,
3. a parabola if $\Delta\neq 0$ and $h^2=ab$,
4. an ellipse if $\Delta\neq 0$ and $h^2<ab$ and
5. a hyperbola if $\Delta\neq 0$ and $h^2>ab$.

Equation of tangent to the conic at $P(x_1, y_1)$

(i) $ax\,x_1+2h\dfrac{(xy_1+x_1y)}{2}+byy_1+g(x+x_1)+f(y+y_1)+c=0$

or $(x-x_1)\ (ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0$

(ii) The equation of normal at the point (x_1, y_1) to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}$$

Note :

To find the equation of tangent at $P(x_1,\ y_1)$ to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ … (*)

We use the following steps :

We replace the terms containing

(i) x^2 and y^2 by $x\,x_1$ and $y\,y_1$ respectively

(ii) xy by $\left(\frac{x\,y_1+x_1y}{2}\right)$

(iii) x and y by $\left(\frac{x+x_1}{2}\right)$ and $\left(\frac{y+y_1}{2}\right)$ respectively in equation (*)

Parabola

Chapter 28

PARABOLA

A parabola is the locus of a point which moves in a plane in such a way that its distance from a fixed point is always equal to its distance from a fixed straight line.

This fixed point is called *focus* and the fixed straight line is called directrix.

Let S be the focus, ZZ' be the directrix and P be any point on the parabola. Then by definition,

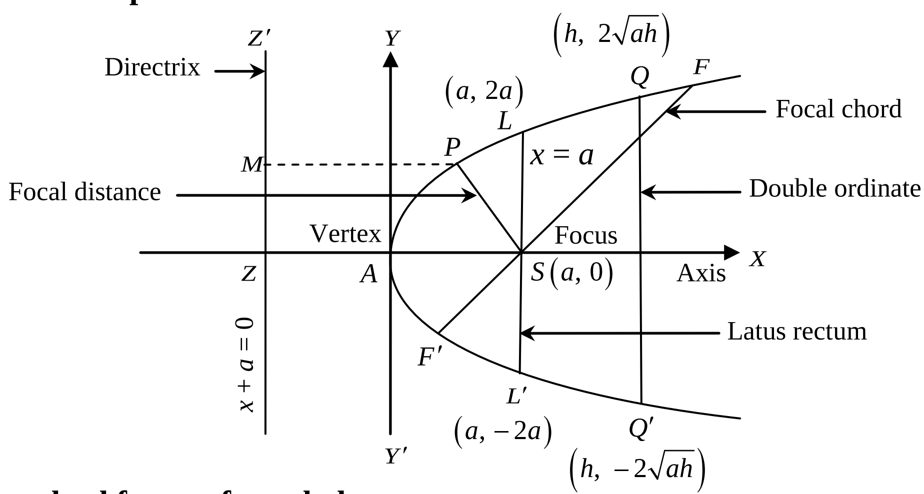
PS = PM

where PM is the length of the perpendicular from P on the directrix ZM.

STANDARD EQUATION OF THE PARABOLA

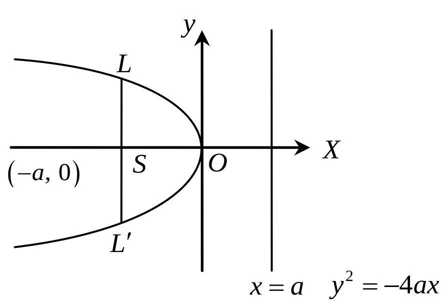
Let S be the focus, zz' be the directrix of the parabola and (x, y) be any point on parabola, then standard form of the parabola is $y^2 = 4ax$.

Some terms related to parabola

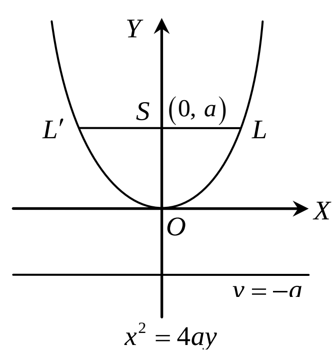


Some other standard forms of parabola are

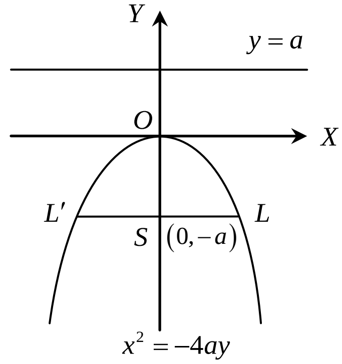
(i) Parabola opening to left i.e. $y^2 = -4ax$



(ii) Parabola opening upwards i.e. $x^2 = 4ay$,



(iii) Parabola opening downwards i.e., $x^2 = -4ay$

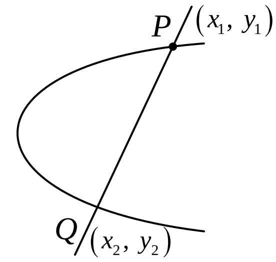


Distance of a point $P(x_1, y_1)$ from focus of the parabola $y^2 = 4ax$ is $= a + x_1$

EQUATION OF A CHORD

1. The equation of chord joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the parabola $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1 y_2$$

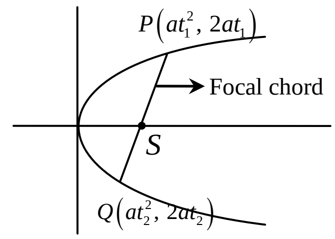


2. The equation of chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$y(t_1 + t_2) = 2(x + at_1 t_2)$$

CONDITION FOR THE CHORD TO BE A FOCAL CHORD

The chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus if $t_1 t_2 = -1$.



LENGTH OF FOCAL CHORD

The length of focal chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$PQ = a(t_2 - t_1)^2.$$

CONDITION OF TANGENCY AND POINT OF CONTACT

The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \dfrac{a}{m}$ and the coordinates of the point of contact are

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

EQUATION OF TANGENT IN DIFFERENT FORMS

1. **POINT FORM**

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$$yy_1 = 2a(x + x_1).$$

2. **PARAMETRIC FORM**

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$ty = x + at^2.$$

3. **SLOPE FORM**

The equation of tangent to parabola $y^2 = 4ax$ in terms of slope 'm' is

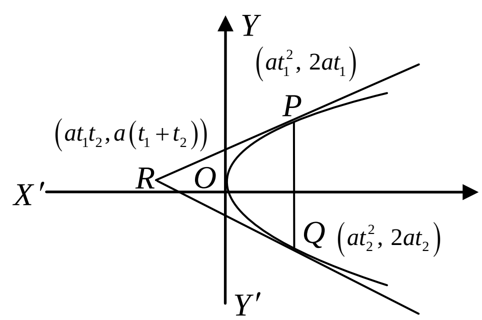
$$y = mx + \frac{a}{m}.$$

The coordinates of the point of contact are $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

POINT OF INTESECTION OF TANGENTS

The point of intersection of tangents drawn at two different points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$R(at_1 t_2, a(t_1 + t_2)).$$



EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **POINT FORM**

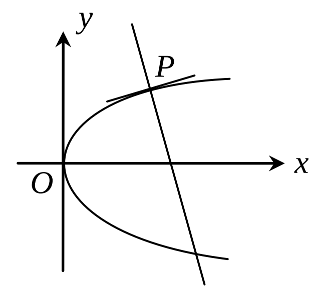
The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

2. **PARAMETRIC FORM**

The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$



3. **SLOPE FORM**

The equation of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx - 2am - am^3.$$

The coordinates of the foot of normal P are $(am^2, -2am)$.

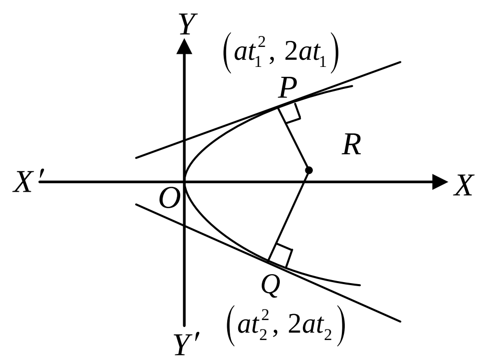
CONDITION FOR NORMALITY

The line $y = mx + c$ is a normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

POINT OF INTERSECTION OF NORMALS

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$R \equiv \left[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)\right]$$



Note :

(a) If the normal at the point 't_1' cuts the parabola again at the point 't_2' then $t_2 = -t_1 - \dfrac{2}{t_1}$

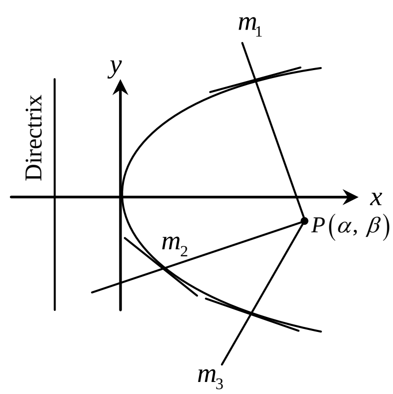
(b) If normals at t_1 and t_2 intersect on the parabola then $t_1t_2 = 2$.

CO-NORMAL POINTS

Any three points on a parabola normals at which pass through a common point are called co-normal points.

If three normals are drawn through a point $P(h, k)$, then their slopes are the roots of the cubic equation :

$$k = mh - 2am - am^3$$



Note :

(a) $m_1 + m_2 + m_3 = 0$ *i.e.*, the sum of the slopes of the normals at co-normals points is zero.

(b) The sum of the ordinates of the co-normal points ($i.e.$, $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$) is zero.

(c) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normals points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ $\therefore$ y-coordinate of the centroid $= \dfrac{-2a(m_1 + m_2 + m_3)}{3} = \dfrac{-2a}{3} \times 0 = 0$. Hence, the centroid lies on the x-axis i.e. axis of the parabola].

(d) If three normals drawn to any parabola $y^2 = 4ax$ from a given point $(h, 0)$ be real, then $h > 2a$.

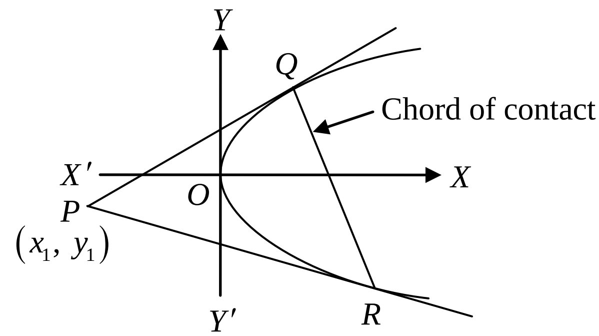
POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, = \text{or} < 0$.

EQUATION OF THE CHORD OF CONTACT OF TANGENTS TO A PARABOLA

Let PQ and PR be tangents to the parabola $y^2 = 4ax$ drawn from any external point $P(x_1, y_1)$ then QR is called the chord of contact of the parabola $y^2 = 4ax$

The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

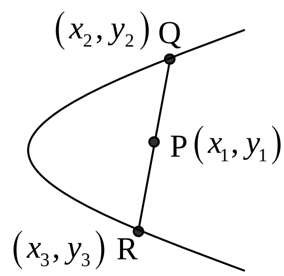


EQUATION OF THE CHORD OF THE PARABOLA WHICH IS BISECTED AT A GIVEN POINT

The equation of the chord of the parabola $y^2 = 4ax$ bisected at the point (x_1, y_1) is given by $T = S_1$ where

$T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$

i.e. $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

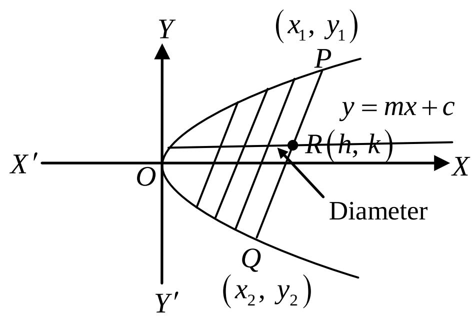


DIAMETERS OF A PARABOLA

The locus of the middle points of a system of parallel chords is called a diameter of the parabola. The diameter is a straight line parallel to the axis of the parabola.

The equation of the diameter bisecting chords of the parabola $y^2 = 4ax$ of slope m is $y = 2a/m$.

Tangents drawn at the ends of any of these chords meet on the diameter of these chords.



POLE AND POLAR

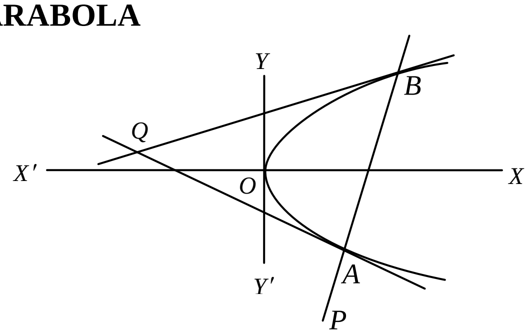
Let P be a given point. Let a line through P intersect the parabola at two points A and B. Let the tangent at A and B intersect at Q. The locus of point Q is a straight line called the polar of point P with respect to the parabola and the point P is called the pole of the polar.

EQUTION OF POLAR OF A POINT WITH RESPECT TO A PARABOLA

The polar of a point $P(x_1, y_1)$ with respect to the parabola

$y^2 = 4ax$ is $T = 0$

where $T \equiv yy_1 - 2a(x + x_1)$.



CONJUGATE POINTS

If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of Q will pass through P and such points are said to be **conjugate points**.

CONJUGATE LINES

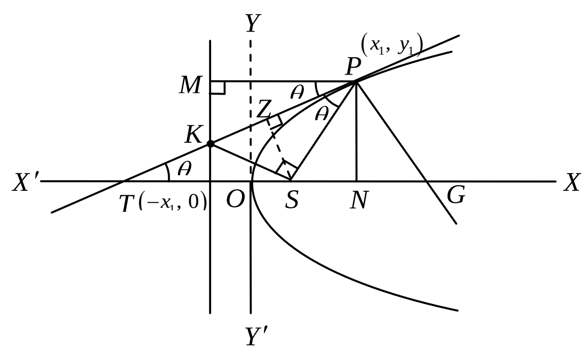
If the pole of line $ax+by+c=0$ lies on the another line $a_1x+b_1y+c_1=0$ then the pole of the second line will lie on the first and such lines are said to be **conjugate lines**.

Note :

(a) Polar of the focus is the directrix.
(b) Any tangent is the polar of its point of contact.
(c) The point of intersection of the polars of two points Q and R is the pole of QR.

SOME MORE IMPORTANT FACTS ABOUT PARABOLA

1. The *parametric equations* of the parabola or the coordinates of any point on it are $x = at^2,\ y = 2at$.
2. The tangents at the extremities of any focal chord intersect at right angles on the directrix.
3. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
4. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.
5. The circle described on any focal chord of a parabola as diameter touches the directrix.
6. If the normal at the point $(at_1^2, 2at_1)$ meets the parabola again at $(at_2^2, 2at_2)$ then $t_2 = -t_1 - 2/t_1$.
7. Three normals can be drawn from a point (x_1, y_1) to the parabola. The points where these normals meet the parabola are called *feet of the normals* or *conormal points*. The sum of the slopes of these normals is zero and the sum of the ordinates of the feet of these normals is also zero.
8. If the normals 't_1' and 't_2' meet on the parabola then $t_1\ t_2 = 2$.
9. The pole of any focal chord of the parabola lies on its directrix.
10. A diameter of the parabola is parallel to its axis and the tangent at the point where it meets the parabola is parallel to the system of chords bisected by the diameter and tangent at the ends of any of the parallel chords of this diameter meet on the diameter.
11. The harmonic mean between the focal radii of any focal chord of a parabola is equal to semi-latus rectum.
12. If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, the
   (i) $ST = SG = SP$, S being the focus
   (ii) $\underline{|PSK} = \pi/2$, where K is the point where the tangent at P meets the directrix.
   (iii) The tangent at P is equally inclined to the axis of the parabola and the focal distance of P.
   (iv) length of the subtangent is twice the abscissa at the point of the tangency for the parabola $y^2 = 4ax$



   (v) length of the sub-normal is always of constant length and is equal to semi latus rectum of the parabola i.e. $2a$
13. If we draw a circle taking any focal radius as diameter will touch the tangent at the vertex.
14. The foot of $\perp$ from focus on any tangent to the parabola is the point where the tangent meets the tangent at vertex.

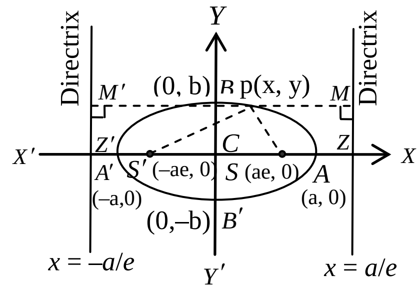
   i.e. here Z lies on y-axis and $SZ = \sqrt{OS.SP}$

Ellipse **Chapter 29**

DEFINITION

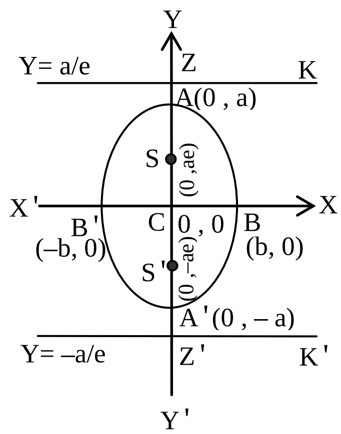
An ellipse is the locus of a point which moves in such a way that its distance from a fixed point is in constant ratio (<1) to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity** of the ellipse, denoted by *e*.

STANDARD EQUATION OF THE ELLIPSE



Let S be the focus, ZM be the directrix of the ellipse and $P(x, y)$ is any point on the ellipse, then by definition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ where } \quad b^2=a^2\left(1-e^2\right)$$



The other form of equation of ellipse is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

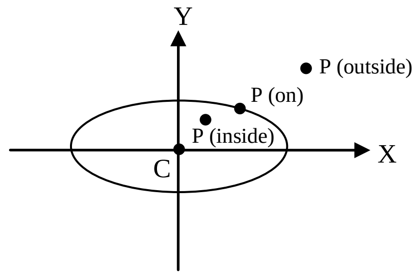
where $b^2=a^2\left(1-e^2\right)$

PARAMETRIC FORM OF THE ELLIPSE

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the equation of ellipse in the parametric form will be given by $x=a\cos\theta,\ y=b\sin\theta$, where θ is the eccentric angle which is such that $0\le\theta<2\pi$. Therefore, coordinate of any point P on the ellipse will be given by $\left(a\cos\theta,\ b\sin\theta\right)$.

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE



Let $P\left(x_1,\ y_1\right)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse.

The point lies outside, on or inside the ellipse if $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>,=,<0$.

INTERSECTION OF A LINE AND AN ELLIPSE

The line $y=mx+c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two distinct point if $a^2m^2+b^2>c^2$, in one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$.

Distance of any point $P\left(x_1,\ y_1\right)$ from focus $F_1=a-ex_1$

Distance of any point $P\left(x_1,\ y_1\right)$ focus $F_2=a+ex_1$

EQUATION OF TANGENT IN DIFFERENT FORMS

1. **Point form:** The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

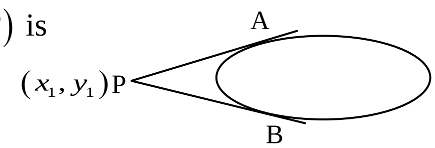
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

2. **Slope form:** If the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c^2 = a^2m^2 + b^2$.

   Hence, the straight line $y = mx \pm \sqrt{a^2m^2+b^2}$ always represents the tangent to the ellipse.

3. **Parametric form :** The equation of tangent at any point $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$



Equation of normal in different forms

1. **Point form:** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

2. **Parametric form :** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$ is

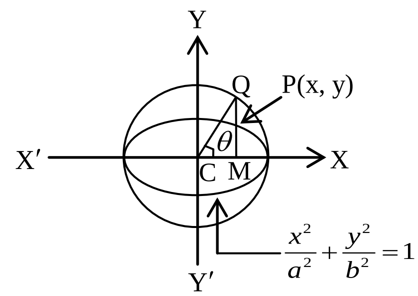
$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2.$$

3. **Slope form :** If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of normal is

$$y = mx \pm \frac{m\left(a^2-b^2\right)}{\sqrt{a^2+b^2m^2}}.$$

AUXILIARY CIRCLE

The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse then its auxiliary circle is $x^2+y^2=a^2$.



ECCENTRIC ANGLE OF A POINT

Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q. Join CQ. The angle $\angle XCQ = \theta$ is called the eccentric angle of the point P on the ellipse.

Note that the angle $\angle XCP$ is not the eccentric angle of point P.

EQUATION OF PAIR OF TANGENTS

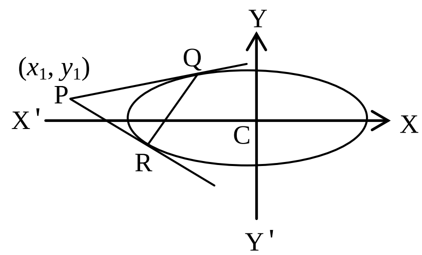
The combined equation of pair of tangents PA and PB is given by $SS_1 = T^2$

where $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1,\quad S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1,\quad T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

DIRECTOR CIRCLE

The director circle is the locus of point from which perpendicular tangents are drawn to the ellipse. Hence locus of $P(x_{1,}y_1)$ *i.e.*, equation of director circle is $x^2+y^2=a^2+b^2$.

CHORD OF CONTACT

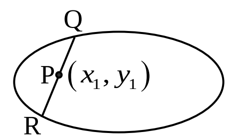


If PQ and PR be the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then QR is called chord of contact and the equation of the chord of contact QR is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ for (x_1, y_1).

EQUATION OF CHORD WITH MID POINT (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose mid point be (x_1, y_1) is given by $T=S_1$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1,\quad S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1.$

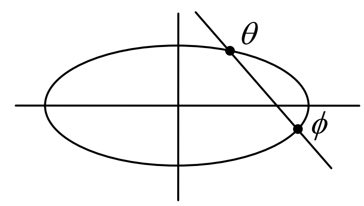


EQUATION OF THE CHORD JOINING TWO POINTS ON AN ELLIPSE

Equation of the chord joining two points on the ellipse having eccentric angles θ and ϕ is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

The point of intersection of tangent draw at the points $P(\theta)$ and $Q(\phi)$ are to the



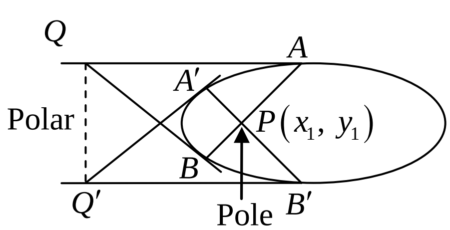
ellipse $\frac{x^2}{a}+\frac{y^2}{b^2}=1$ are $\left(a\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, b\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

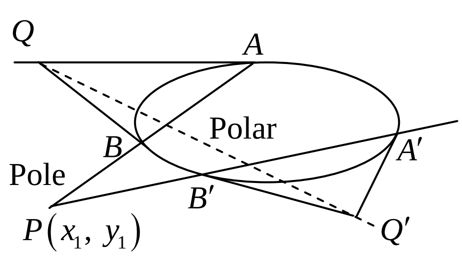
The point of intersection of normal draw at the $P(\theta)$ and $Q(\phi)$ are to the ellipse $\frac{x^2}{a}+\frac{y^2}{b^2}=1$ are

$$\left(\frac{a^2-b^2}{a}\cos\theta\cos\phi\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\ -\frac{a^2-b^2}{b}\sin\theta\sin\phi\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$$

POLE AND POLAR

Let $P(x_1,\ y_1)$ be any point inside or outside the ellipse. A chord through P intersects the ellipse at A and B respectively. If tangents to the ellipse at A and B meet at Q then locus of Q is called polar of P will respect to ellipse and point P is called pole.





Equation of polar : Equation of polar of the point (x_1, y_1) with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \text{ i.e., } T=0.$$

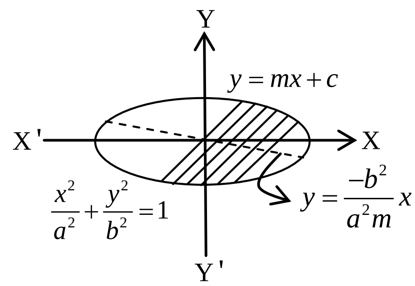
PROPERTIES OF POLE AND POLAR

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ passes through $P(x_1, y_1)$ and such points are said to be **conjugate points**.
2. If the pole of a line $l_1x+m_1y+n_1=0$ lies on the another line $l_2x+m_2y+n_2=0$, then the pole of the second line will be on the first and such lines are said to be **conjugate lines**.
3. Pole of a given line is same as point of intersection of tangents at its extremities.

DIAMETER OF THE ELLIPSE

Definition : A line through the center of an ellipse is called a diameter of the ellipse.

The equation of the diameter bisecting the chords $y=mx+c$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$, which is passing through (0, 0)

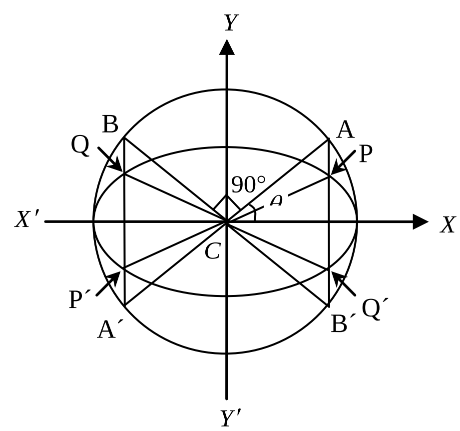


Conjugate diameter : Two diameter of an ellipse are said to be conjugate diameter if each bisects all chords parallel to the other. The coordinates of the four extremities of two conjugate diameters are

$$P(a\cos\theta, b\sin\theta); P'(-a\cos\theta, -b\sin\theta),$$
$$Q(-a\sin\theta, b\cos\theta); Q'(a\sin\theta, -b\cos\theta)$$

If $y=m_1x$ and $y=m_2x$ be two conjugate diameters of an ellipse, then

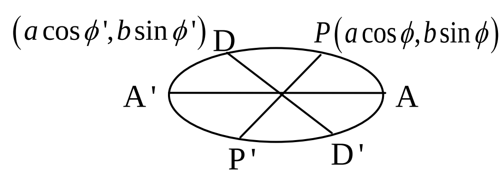
$$m_1m_2=\frac{-b^2}{a^2}.$$



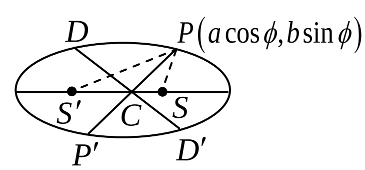
1. **Properties of diameters :**
   (i) The tangents at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter ,
   (ii) The tangents at the ends of any chord meet on the diameter which bisects the chord.

2. **Properties of conjugate diameters :**
   (i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle,

   i.e., $\phi'-\phi=\frac{\pi}{2}.$



   (ii) The sum of the squares of any two conjugate semidiameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse *i.e.,* $CP^2+CD^2=a^2+b^2$.



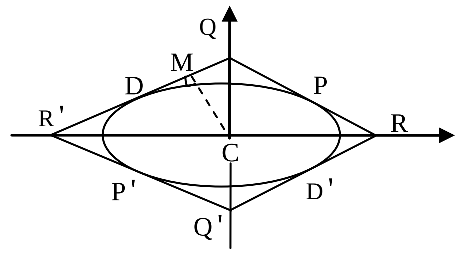
(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi diameter which is conjugate to the diameter through the point

i.e. $SP.S'P = CD^2$.

(iv) The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to product of the axes. *i.e.*

Area of parallelogram = (2*a*) (2b)

= Area of rectangle contained under major and minor axes.



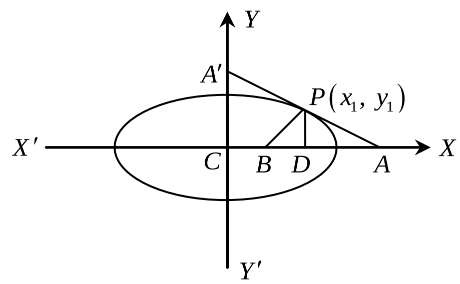
3. **Equi-conjugate diameters:** Two conjugate diameters are called equi-conjugate, if their lengths are equal *i.e.*, $(CP)^2 = (CD)^2$.

$\therefore \quad CP = CD = \sqrt{\dfrac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

SUBTANGENT AND SUBNORMAL

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively. Length of subtangent at $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

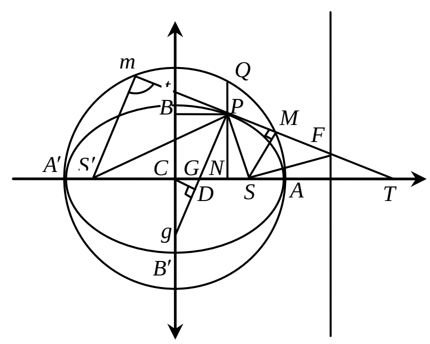
$$DA = CA - CD = \frac{a^2}{x_1} - x_1.$$



Length of sub-normal at $P(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$BD = CD - CB = x_1 - \left(x_1 - \frac{b^2}{a^2}x_1\right) = \frac{b^2}{a^2}x_1 = (1-e^2)x_1.$$

PROPERTIES OF THE ELLIPSE



1. $\dfrac{PN^2}{AN \cdot A'N} = \dfrac{b^2}{a^2}$

   (i) $AS \cdot A'S = b^2$

   (ii) $CN \cdot CT = a^2$

   (iii) $Cn \cdot Ct = b^2$

2. The locus of the feet of the perpendiculars M and m from the foci on any tangent to ellipse is its auxiliary circle

   The product of the two perpendicular distances from the foci on any tangent of a ellipse is equals to b^2 *i.e.* $SM \cdot S'm' = b^2$.

3. $CG = e^2.\ CN = e^2 x_1$.

4. The tangent and normal at a point on an ellipse bisect the angles between the focal radii to the point

   $SG = e.\ SP, \quad S'G = e.\ S'P.$

5. If the tangent on an ellipse meets the directrix in F, then PF subtends a right angle at the corresponding focus i.e. $\angle PSF = 90°$

6. Tangents at the ends of a focal chord intersect on the directrix.

6. If the normal at any point P meets the major and minor axes in G and g and CD is the perpendicular upon the normal, then $PD \times PG = b^2$ and $PD \times Pg = a^2$.
7. Tangents at the ends of any chord meet on the diameter which bisects the chord.
8. The sum of distances of any point P on the ellipse from the focus S and S' is $2a$ i.e. $PS + PS' = 2a$
9. The ratio of y-coordinates of corresponding points on ellipse and Auxiliary circle $= b : a$
10. The Harmonic mean of focal radii of any focal chord is equal to semi-latus rectum $= \dfrac{b^2}{a}$.
11. If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four concyclic points on an ellipse then

    $\alpha + \beta + \gamma + \delta = 2n\pi,\ n \in I.$
12. If eccentric angles of feet P, Q, R, S of these normals be $\alpha, \beta, \gamma, \delta$ then

    $\alpha + \beta + \gamma + \delta = (2n+1)\pi,\ n \in I$
13. The necessary and sufficient condition for the normals at three α, β, γ points on the ellipse to be concurrent if $\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$

Hyperbola

Chapter 30

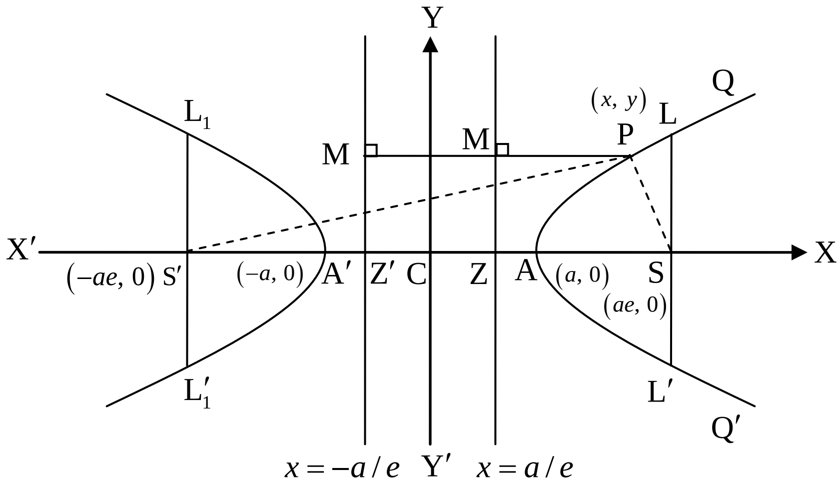
DEFINITION

A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is greater than unity.

STANDARD EQUATION OF THE HYPERBOLA

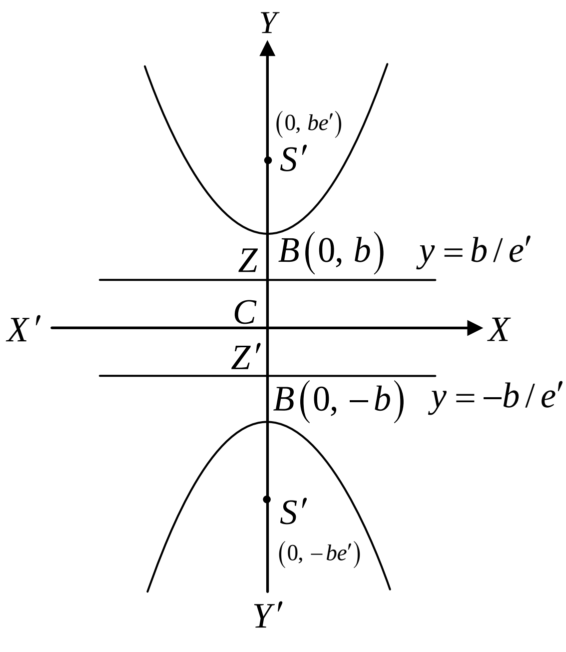
Let S be the focus, ZM be the directrix and e be the eccentricity of the hyperbola, then by definition,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ where } b^2=a^2\left(e^2-1\right).$$



CONJUGATE HYPERBOLA

This hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of the given hyperbola.



Equation of conjugate hyperbola of a given hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

Difference between both hyperbola will be clear from the following table :

Imp. terms \ Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	$(0, 0)$	$(0, 0)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be')$
Equation of directrices	$x = \pm\, a/e$	$y = \pm\, b/e'$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e' = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of latus rectum	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a\sec\phi, b\tan\phi)$ $0 \le \phi < 2\pi$	$(a\tan\phi, b\sec\phi)$ $0 \le \phi < 2\pi$
Focal radii	(a) If P lies on right branch $SP = ex_1 - a$, $S'P = ex_1 + a$ (b) If P lies on left branch $SP = -ex_1 + a$, $S'P = -ex_1 - a$	(a) If P lies on upper branch $SP = e'y_1 - b$, $S'P = e'y_1 + b$ (b) If P lies on lower branch $SP = -e'y_1 + b$, $S'P = -e'y_1 - b$
Difference of focal radii $\lvert S'P - SP\rvert$	$2a$	$2b$
Tangent at the vertices	$x = -a,\ x = a$	$y = -b,\ y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$

Note :

(a) If e, e' are eccentricity of hyperbola & conjugate hyperbola then $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

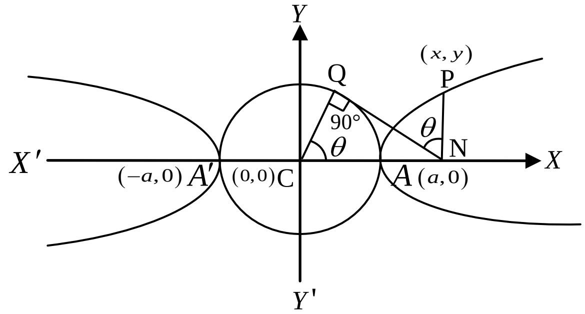
(b) Foci of a hyperbola & conjugate hyperbola are concyclic.
i.e. $ae = be'$

AUXILIARY CIRCLE OF HYPERBOLA

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the hyperbola, than equation of the auxiliary circle is

$$x^2 + y^2 = a^2.$$

Let $\angle QCN = \theta$. Here P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \theta < 2\pi)$.



PARAMETRIC EQUATIONS OF HYPERBOLA

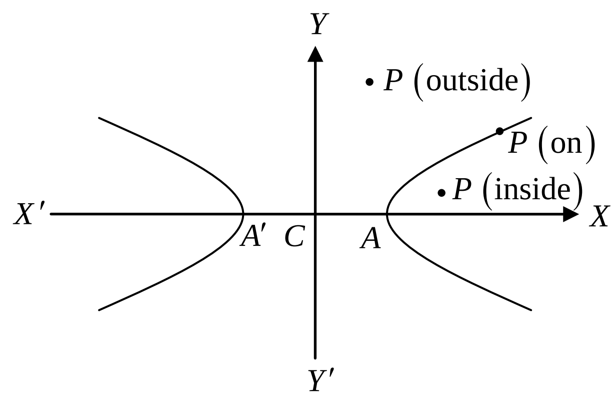
The equation $x = a\sec\theta$ and $y = b\tan\theta$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Thus $(a\sec\theta, b\tan\theta)$ lies on the hyperbola for all values of θ.

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Then $P(x_1, y_1)$ will lie inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is positive, zero or negative.



i.e. $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0 \Rightarrow$ inside

$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1=0 \Rightarrow$ on

$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0 \Rightarrow$ outside

INTERSECTION OF A LINE AND A HYPERBOLA

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

Condition of tangency : If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2 = a^2m^2 - b^2$.

EQUATIONS OF TANGENT IN DIFFERENT FORMS

1. **Point form** : The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

2. **Parametric form** : The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

3. **Slope form :** The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

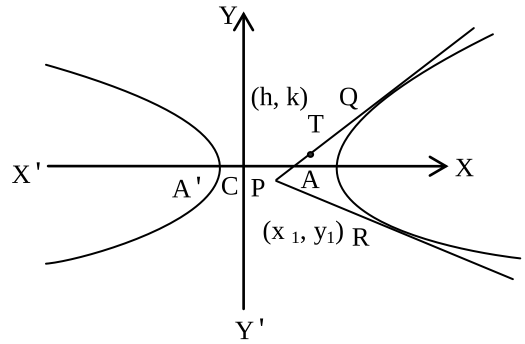
The coordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right).$$

EQUATION OF PAIR OF TANGENTS

If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then a pair of tangents PQ, PR can be drawn to it from P. The equation of pair of tangents PQ and PR is $SS_1 = T^2$ where

$$S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1, S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1, T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$$



DIRECTOR CIRCLE

The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$x^2+y^2=a^2-b^2.$$

EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **Point form : The equation of normal to the hyperbola**

   $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2.$

   The equation of normal at the point $(x_1,\ y_1)$ to the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ is

   $$\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}$$

2. **Parametric form:** The equation of normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

   $$ax\cos\theta+\text{by}\cot\theta = a^2+b^2$$

3. **Slope form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m of normal is

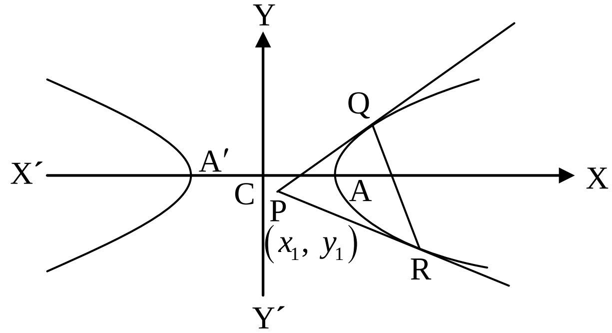
   $$y=mx\pm\frac{m\left(a^2+b^2\right)}{\sqrt{a^2-b^2m^2}}$$

4. **Condition for normality :** If $y=mx+c$ is the normal of

   $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$ then $c=\pm\frac{m\left(a^2+b^2\right)}{\sqrt{a^2-m^2b^2}}$ or $c^2=\frac{m^2\left(a^2+b^2\right)^2}{\left(a^2-m^2b^2\right)},$ which is condition of normality

Equation of chord of contact of tangents drawn from a point to a hyperbola

Let PQ and PR be tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ drawn from any external point $P(x_1, y_1)$. Then equation of chord of contact QR is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T=0$

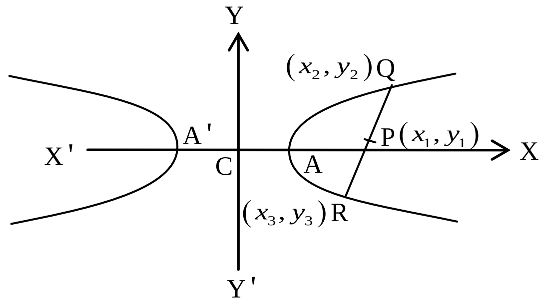


Equation of the chord of the hyperbola whose mid point (x_1, y_1) is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the given point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

i.e. $T = S_1$



Equation of the chord joining two points on the hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

Point of intersection of tangents

Point of intersection of tangents at $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\left(a\frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)},\ b\frac{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right)$$

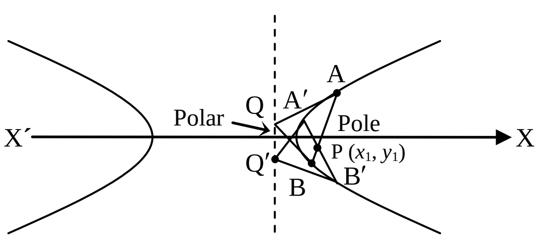
Point of intersection of normals

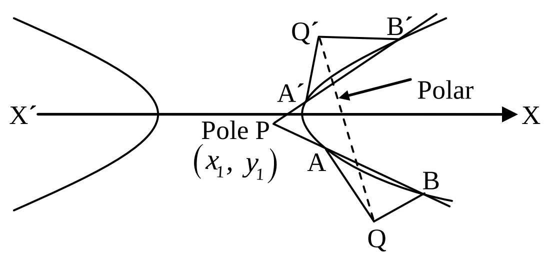
Coordinates of point of intersection of normals at $P(a\sec\phi_1, b\tan\phi_1)$ and $Q(a\sec\phi_2, b\tan\phi_2)$ is

$$\left(\frac{a^2+b^2}{a}\sec\phi_1\sec\phi_2\frac{\cos\left(\frac{\phi_1-\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)},\ -\frac{a^2+b^2}{b}\tan\phi_1\tan\phi_2\frac{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}\right).$$

POLE AND POLAR

The locus of the point of intersection of the tangents at the end of a variable chord drawn from a fixed point P on the hyperbola is called the polar of the given point P with respect to the hyperbola and the point P is called the pole of the polar. The equation of the required polar with (x_1, y_1) as its pole is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.





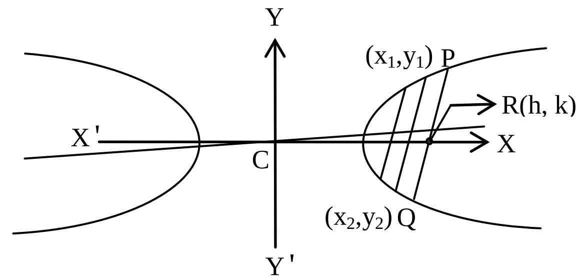
PROPERTIES OF POLE AND POLAR

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be **conjugate points**.

2. If the pole of a line $lx+my+n=0$ lies on the another line $l_1x+m_1y+n_1=0$ then pole of the second line will lie on the first and such lines are said to be **conjugate lines**.
3. Pole of a given line is same as point of intersection of tangents as its extremities.

DIAMETER OF THE HYPERBOLA

The locus of the middle point of a system of parallel chords of a hyperbola is called a diameter and the point where the diameter intersects the hyperbola is called the vertex of the diameter.



Let $y=mx+c$ system of parallel chords to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for different chords then the equation of diameter of the hyperbola is $y=\frac{b^2x}{a^2m}$, which is passing through (0, 0).

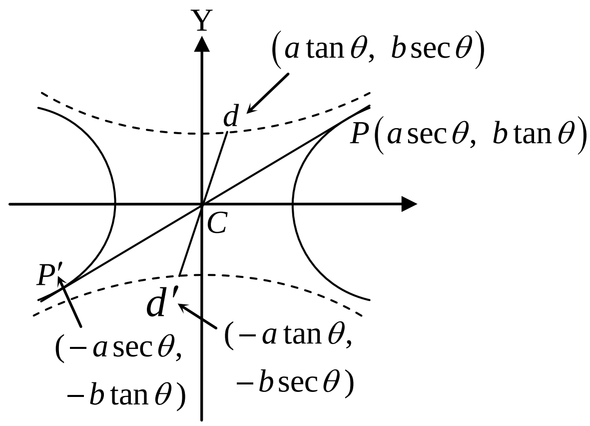
Conjugate diameter : Two diameter are said to be conjugate when each bisects all chords parallel to the other.

If $y=m_1x,\ y=m_2x$ be conjugates, diameters, then $m_1m_2=\frac{b^2}{a^2}$.

Note : If the two extremities of a diameter lie in the first and third quadrants, the extremities of the conjugate diameter also lie in the first and third quadrants.

The coordinates of the four extremities of two conjugate diameters are shown in the adjoining figure.

Caution : The extremities d and d' of the conjugate diameter do not lie on the hyperbola.



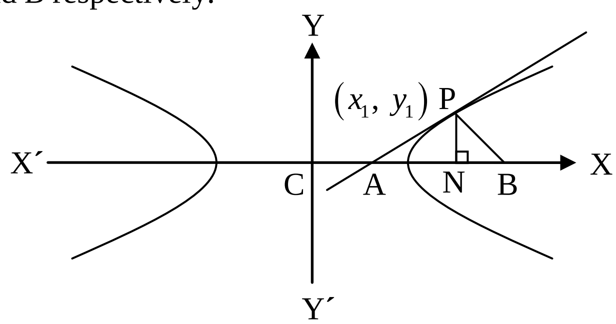
Subtangent and Subnormal of the hyperbola

Let the tangent and normal at $P(x_1,\ y_1)$ meet the x-axis at A and B respectively.

Length of subtangent $AN=CN-CA=x_1-\frac{a^2}{x}$.

Length of subnormal $BN=CB-CN=\frac{(a^2+b^2)}{a^2}x_1-x_1$

$=\frac{b^2}{a^2}x_1=(e^2-1)x_1$.



ASYMPTOTES OF A HYPERBOLA

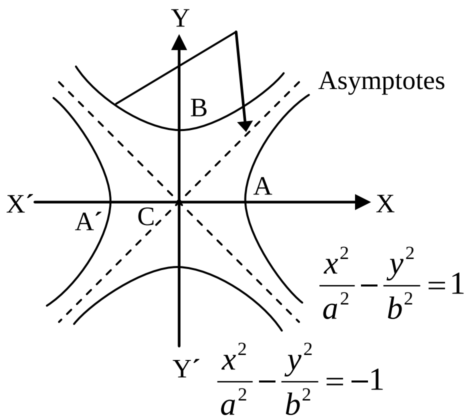
If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called an **asymptote of the hyperbola**.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $y=\pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$.

SOME IMPORTANT POINTS ABOUT ASYMPTOTES

1. The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$.

2. When $b = a$ i.e., the asymptote of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.
3. A hyperbola and its conjugate hyperbola have the same asymptotes.



4. The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only i.e., Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola or

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1\right)$$

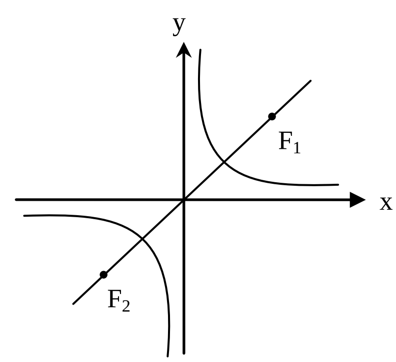
5. The asymptotes pass through the centre of the hyperbola.
6. The bisector of the angles between the asymptotes are the coordinate axes.
7. The angle between the asymptotes of the hyperbola $S = 0$ i.e., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\frac{b}{a}$ or $2\sec^{-1} e$.
8. Asymptotes are equally inclined to the axes of the hyperbola.

RECTANGULAR OR EQUILATERAL HYPERBOLA

1. **Definition :** A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$.

   The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of $x^2 + \text{coefficiant of } y^2 = 0$
2. Rotating the axes by an angle $-\pi/4$ about the same origin the equation of rectangular hyperbola $x^2 - y^2 = a^2$ reduces to $xy = c^2\left(= \frac{a^2}{2}\right)$.
3. **Parametric co-ordinates of a point on the hyperbola xy = c^2 If t is non zero variable , the co ordinates of any point on the rectangular** hyperbola xy = c^2 can be written as (ct, c/t). The point (ct, c/t) on the hyperbola $xy = c^2$ is generally referred as the point 't'.

   For rectangular hyperbola $xy = c^2$, the coordinates of foci are $\left(c\sqrt{2}, c\sqrt{2}\right)$ and $\left(-c\sqrt{2}, -c\sqrt{2}\right)$ directrices are $x + y = \pm c\sqrt{2}$.



4. **Equation of the chord joining points t_1 and t_2 :** The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2 \Rightarrow x + y\,t_1 t_2 = c\left(t_1 + t_2\right)$.

5. **Equation of the tangents in different forms**

(i) Point form: The equation of tangent at (x_1, y_1) to the hyperbola

$$xy = c^2 \text{ is } xy_1 + yx_1 = 2c^2 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2.$$

(ii) Parametric form: The equation of the tangents at $\left(ct, \frac{c}{t}\right)$ to the hyperbola

$$xy = c^2 \text{ is } \frac{x}{t} + yt = 2c \;\Rightarrow\; x + t^2 y = 2ct$$

On replacing x_1 by ct and y_1 by $\frac{c}{t}$ in the equation of the tangents at (x_1, y_1)

i.e. $xy_1 + yx_1 = 2c^2$ we get $\frac{x}{t} + yt = 2c$.

Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

6. **Equation of the normal in different forms:**

(i) Point form: The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(ii) Parametric form: The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola

$xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$

This equation is a fourth degree in t. So, in general four normals can be drawn from a point to the hyperbola $xy = c^2$, and point of intersection of normals at t_1 and t_2 is

$$\left(\frac{c\left\{t_1t_2\left(t_1^2 + t_1t_2 + t_2^2\right) + 1\right\}}{t_1t_2(t_1+t_2)}, \frac{c\left\{t_1^3t_2^3 + \left(t_1^2 + t_1t_2 + t_2^2\right)\right\}}{t_1t_2(t_1+t_2)}\right).$$

(iii) If the normal at $P\left(ct, \frac{c}{t}\right)$ cuts the rectangular hyperbola $xy = c^2$ at $Q\left(ct', \frac{c}{t'}\right)$ then $t' = -\frac{1}{t^3}$.

7. Equation of diameter of rectangular hyperbola $xy = c^2$ is $y + mx = 0$ (m is the slope of the chord joining two points lies on the rectangular hyperbola)

Two diameters $y + m_1x = 0$ and $y + m_2x = 0$ are conjugate diameter if $m_1 + m_2 = 0$.

PROPERTIES OF HYPERBOLA $x^2/a^2 - y^2/b^2 = 1$

1. If *PN* be the ordinate of a point *P* on the hyperbola and the tangent at *P* meets the transverse axis in *T*, then $ON.OT = a^2$, O being the origin.
2. If PM be drawn perpendiculars to the conjugate axis from a point *p* on the hyperbola and the tangent at P meets the conjugate axis in T, then $OM.OT = -b^2$; O, being the origin.
3. If the normal at *P* on the hyperbola meets the transverse axis in *G*, then $SG = eSP$; S being a foci and e the eccentricity of the hyperbola.
4. The tangent and normal at any point of a hyperbola bisect the angle between the focal radii to that point.
5. The locus of the feet of the perpendiculars from the foci on a tangent to a hyperbola is the auxiliary circle.
6. The product of the length of the perpendicular drawn from foci on any tangent to hyperbola is b^2.

7. From any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ perpendicular are drawn to the asymptotes then product is $\frac{a^2b^2}{a^2+b^2}$ and for rectangular hyperbola $=\frac{a^2}{2}$
8. If a circle cuts the rectangular hyperbola $xy=1$ in $(x_r,\ y_r)$ (four points) $r=1,\ 2,\ 3,\ 4$ then $\frac{1}{x_1x_2x_3x_4}=y_1y_2y_3y_4=1$.
9. A rectangular hyperbola with centre at C is cut by any circle of radius R in four points L, M, N, P then the value of $CL^2+CM^2+CN^2+CP^2=4R^2$.
10. If a triangle is inscribed in a rectangular hyperbola then the orthocenter of triangle lies on the rectangular hyperbola.
11. The portion of tangent intercepted between the asymptotes at any point of the hyperbola is bisected by the point of contact.
12. Whenever any circle and any hyperbola cut each other at four points the mean position of these four points is the mid point of the line segment joining centre of hyperbola and centre of circle.
13. The harmonic mean of focal radi for any focal chord $=\frac{b^2}{a}$
14. Tangent drawn at the ends of any focal chord meet on the directrix.
15. Local of point of intersection two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle called director circle whose equation is $x^2+y^2=a^2-b^2$

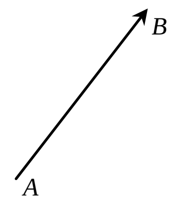
    and if $a<b$ then there is no real point from where we can draw two perpendicular tangents to the hyperbola.
16. The portion of tangent between point of contact and the point where it cuts the directrix subtend $90°$ angle at the focus.

Vectors **Chapter 31**

Physical Quantity : The quantity by means of which we describe the laws of physics are called physical quantities.
Scalars : The physical quantities which have magnitude only are called scalars. These quantities are specified by number and a unit.
Examples : Length, mass, volume, area, temperature, density, work etc.

Vectors : The physical quantities which have magnitude and direction both and also obey the triangle law of addition are called vectors.
Examples : Displacement, velocity, acceleration, force, momentum and couple.

REPRESENTATION AND NOTATION OF A VECTOR
A vector is represented by a directed line segment distinguished by **initial and terminal point.**
The directed line segment with initial point A and terminal point B is denoted by symbol $\overrightarrow{AB}$ or **AB**



MODULUS (OR MAGNITUDE) OF A VECTOR
The positive real number which is the measure of the length of the vector, is called the **modulus, length, magnitude, absolute value** or **norm** of the vector.

The modulus of a vector **a** or ***OA*** is usually denoted by $|\mathbf{a}|$ or $|\overrightarrow{OA}|$ or by the corresponding letter a (not in bold-faced type).

Thus, $|\overrightarrow{OA}| = OA$ and $|\mathbf{a}| = \text{a}$.

MULTIPLICATION OF A VECTOR BY A SCALAR
The product of a scalar m and a vector **a**, is defined as a vector $m\mathbf{a}$ or $\mathbf{a}m$ whose magnitude is the product of the magnitudes of m and **a** and whose direction is that of **a** or opposite to **a** according as m is positive or negative.
EQUAL VECTORS
Two vectors **a** and **b** are equal i.e. $\mathbf{a} = \mathbf{b}$ if and only if they have (*i*) same magnitude and (*ii*) same direction.
UNIT VECTORS
A vector whose magnitude is unity is called a unit vector. A unit vector along the direction of a given vector **a** is usually denoted by the symbol $\hat{\mathbf{a}}$ and read as '**a** cap'.

Then, we have $\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}}$ … (1)

i.e. a vector $=$ its modulus $\times$ unit vector in its direction

Also $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$

i.e. unit vector in a direction $= \dfrac{\text{vector in that direction}}{\text{modulus of vector}}$

ZERO OR NULL VECTOR
A vector whose magnitude is zero, is called a zero vector.

For such a vector, initial and terminal points are coincident so that its direction is indeterminate. A zero vector is denoted by the bold-faced symbol $\mathbf{0}$ as distinguished from the scalar O. Zero vector is also denoted by $\vec{O}$.

COLLINEAR (OR PARALLEL VECTORS)

Vectors which are parallel to the same straight line are called collinear vectors.

Vectors which are not parallel to the same line are called **non-collinear vectors.**

LIKE AND UNLIKE VECTORS

Collinear vectors having the same direction are called **like** vectors and those having the opposite directions are called **unlike** vectors.

AN IMPORTANT PROPERTY OF TWO COLLINEAR (OR PARALLEL) VECTORS

If two vectors **a** and **b** are collinear, then there exists a scalar m such that $\mathbf{b} = m\mathbf{a}$, m being positive or negative according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike vectors

Conversely, if $\mathbf{b} = m\mathbf{a}$ be given, then $\mathbf{a}$ and $\mathbf{b}$ must be collinear (or parallel) vectors such that $|\mathbf{b}| = |m||\mathbf{a}|$.

RECIPROCAL VECTOR

Let a denote the modulus of the given vector $\mathbf{a}$. Then a vector whose direction is that of $\mathbf{a}$ but modulus is $1/|\mathbf{a}|$ (i.e. reciprocal of the modulus of $\mathbf{a}$) is called the reciprocal of $\mathbf{a}$ and is written as $\mathbf{a}^{-1}$.

$$\text{Thus, } \mathbf{a}^{-1} = \frac{1}{|\mathbf{a}|}\hat{\mathbf{a}} = \frac{|\mathbf{a}|}{|\mathbf{a}|^2}\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|^2}.$$

COPLANAR AND NON-COPLANAR VECTORS

Three or more vectors are said to be coplanar when they are parallel to the same plane. Otherwise they are said to be non-coplanar vectors.

CO-INITIAL VECTORS

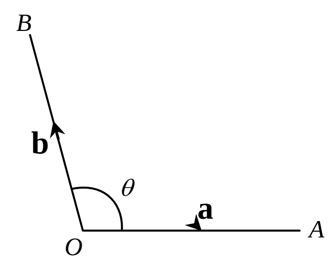
The vectors which have the same initial point are called co-initial vectors.

NEGATIVE OF A VECTOR

A vector having the same modulus as that of a given vector **a** and the direction opposite to that of **a**, is called the negative of $\mathbf{a}$ and is denoted by $-\mathbf{a}$. Clearly, if $\mathbf{OA} = \mathbf{a}$, then $\mathbf{AO} = -\mathbf{a}$, and therefore, $\mathbf{OA} = -\mathbf{AO}$.

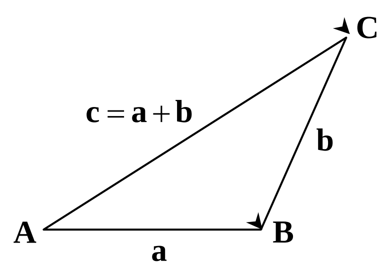
ANGLE BETWEEN TWO VECTORS

The angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ represented by $\mathbf{OA}$ and $\mathbf{OB}$, is defined as the angle AOB which does not exceed π. This is also known as the **inclination of given vectors a** and **b.** If the angle AOB be θ, then $0 \le \theta \le \pi$.



When $\theta = \dfrac{\pi}{2}$, the vectors are said to be **perpendicular** or **orthogonal** and when $\theta = 0$ or π, they are said to be **parallel** or **coincident.**

ADDITION OF VECTOR



(i) Triangle law of addition: If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle from tail of the first vector to the head of the second vector. This is known as the triangle law of vector addition. Thus,

If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, and $\overrightarrow{AC} = \mathbf{c}$

then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e $\mathbf{a} + \mathbf{b} = \mathbf{c}$

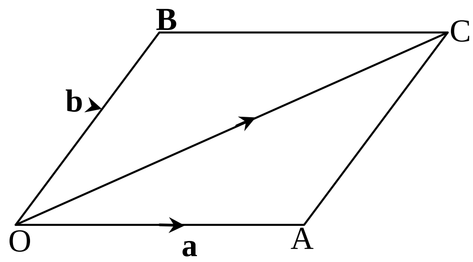
(ii) Parallelogram Law of Addition : If two vector are represented by two adjacent of a parallelogram, then their sum is represented by the diagonal of the parallelogram.

Thus, if $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{OC} = \mathbf{c}$ then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$ where OC is a diagonal of the parallelogram OABC.

(iii) Addition in component from:

If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ their sum is defined as $\mathbf{a}+\mathbf{b} = (a_1+b_1)\hat{\mathbf{i}} + (a_2+b_2)\hat{\mathbf{j}} + (a_3+b_3)\hat{\mathbf{k}}$



PROPERTIES OF VECTOR ADDITION

1. Vector addition is commutative

i.e. for any two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}.$$

2. Vector addition is associative i.e. for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

3. Existence of additive identity

For every vector $\mathbf{a}$, we have

$$\mathbf{a}+\mathbf{0} = \mathbf{a} = \mathbf{0}+\mathbf{a},$$

where $\mathbf{0}$ is the null vector.

4. Existence of additive inverse

Corresponding to a given vector $\mathbf{a}$ there exists a vector $-\mathbf{a}$ such that

$$\mathbf{a}+(-\mathbf{a}) = (-\mathbf{a})+\mathbf{a} = \mathbf{0}.$$

The vector $-\mathbf{a}$ is called the additive inverse of $\mathbf{a}$.

PROPERTIES OF MULTIPLICATION OF VECTOR BY A SCALAR

1. If $m = O$, then $m\mathbf{a} = \mathbf{0}$

2. If m and n be two scalars, then $m(n\mathbf{a}) = mn\mathbf{a} = n(m\mathbf{a})$

3. If m and n be two scalars, then $(m+n)\mathbf{a} = m\mathbf{a}+n\mathbf{a}.$

4. If $\mathbf{a}$, $\mathbf{b}$ are any two vectors and m be any scalar, then $m(\mathbf{a}+\mathbf{b}) = m\mathbf{a}+m\mathbf{b}.$

SUBTRACTION (OR DIFFERENCE) OF TWO VECTORS

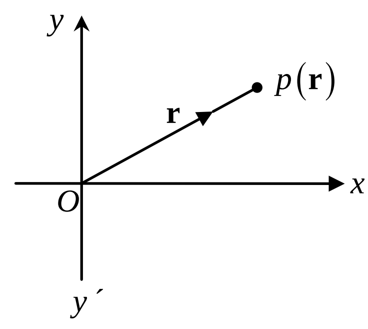
If $\mathbf{a}$ and $\mathbf{b}$ be any two given vectors, then subtraction of $\mathbf{b}$ from $\mathbf{a}$ is defined as the addition of $-\mathbf{b}$ to $\mathbf{a}$.

i.e. $\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b}).$

Hence to subtract a vector $\mathbf{b}$ from $\mathbf{a}$, we should reverse the direction of $\mathbf{b}$ and add to $\mathbf{a}$.

POSITION VECTOR OF A POINT

Let O be the origin and P be any point then vector $\mathbf{r} = \overrightarrow{OP}$ is know as position vector of point P related to O.



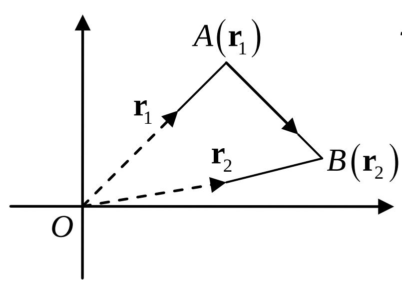
MORE ABOUT THE POSITION VECTORS

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

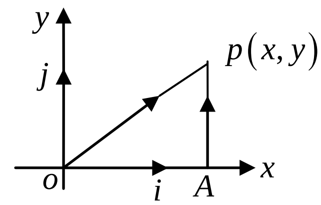
$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= p.v. \text{ of } B - p.v. \text{ of } A = \mathbf{r}_2 - \mathbf{r}_1$$

$$\Rightarrow \quad \overrightarrow{AB} = \mathbf{r}_2 - \mathbf{r}_1$$



COMPONENT OF VECTOR IN TWO DIMENSION



Let $P(x,y)$ be a point in the plane .

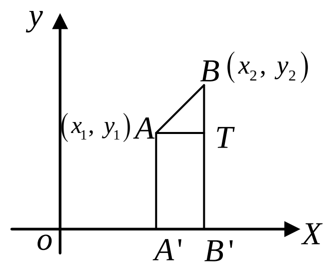
$$\overrightarrow{OA} = x\,\hat{\mathbf{i}} \text{ and } \overrightarrow{AP} = y\,\hat{\mathbf{j}}$$

$$\overrightarrow{OP} = \mathbf{r}$$

then $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

Then $x\,\hat{\mathbf{i}}$ and $y\,\hat{\mathbf{j}}$ are known as component of $\mathbf{r}$ and magnitude of $\mathbf{r}$ *i.e.* $|\mathbf{r}| = \sqrt{x^2+y^2}$ is known as distance between $O(0,0)$ and $P(x,y)$.

Let $A(x_1,y_1), B(x_2,y_2)$ be two points. Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ are unit vectors along OX *and* OY respectively.



$\overrightarrow{AT} = (x_2 - x_1)\hat{\mathbf{i}}$ and $\overrightarrow{TB} = (y_2 - y_1)\hat{\mathbf{j}}$

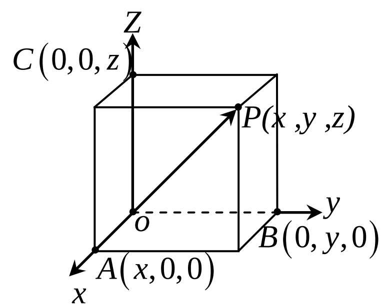
are known as components of vector $\overrightarrow{AB}$ along x–axis and y–axis respectively

$\therefore \quad \overrightarrow{AB} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}}$

and $|\overrightarrow{AB}|$ represent the length of the vector $\overrightarrow{AB}$

$\therefore \quad |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

COMPONENTS OF A VECTOR IN THREE DIMENSION



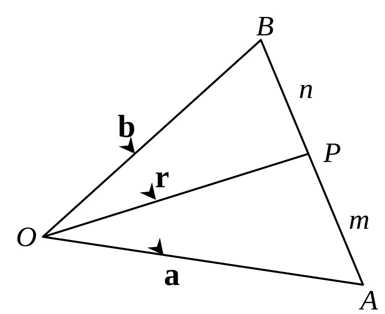
Let $p(x,y,z)$ be a point. $x = OA, y = OB, z = OC$ and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit vectors along *x*-axis , *y*-axis and *z*-axis respectively,

then $\overrightarrow{OA} = x\,\hat{\mathbf{i}}$, $\overrightarrow{OB} = y\,\hat{\mathbf{j}}$, $\overrightarrow{OC} = z\,\hat{\mathbf{k}}$ are components of $\overrightarrow{OP}$ along x, y, z axis respectively

$\therefore \quad \overrightarrow{OP} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $|\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$

Component of a vector $\mathbf{r}$ in the direction of $\mathbf{a}$ is equal to $\dfrac{\mathbf{r}.\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$ and perpendicular to $\mathbf{a} = \mathbf{r} - \dfrac{\mathbf{r}.\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}.$, Projection of $\mathbf{a}$ in the direction of $\mathbf{b} = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{b}|}$.

SECTION FORMULA



1. **INTERNAL DIVISION**

   The position vector of the point P which divides internally the join of two given points A and B whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ in a given ratio $m:n$, is

$$\mathbf{OP} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n}.$$

Note :

(i) If P is the mid point of AB, then it divides AB in the ratio $1:1$. Therefore, P.V. of P is given by

$$\mathbf{OP} = \frac{\mathbf{a}+\mathbf{b}}{2}.$$

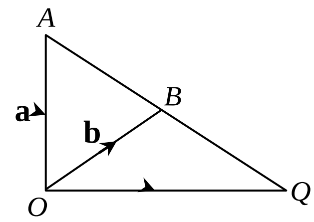
(ii) We have,

$$\mathbf{OP} = \mathbf{r} = \frac{m\mathbf{b} + n\mathbf{a}}{m+n} = \frac{m}{m+n}\mathbf{b} + \frac{n}{m+n}\mathbf{a}$$

$$= \lambda\mathbf{a} + \mu\mathbf{b}, \text{ where } \lambda = \frac{n}{m+n} \text{ and } \mu = \frac{m}{m+n}.$$

Thus, p.v. of any point P on AB can always be taken as $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}$, where $\lambda + \mu = 1$.

2. **EXTERNAL DIVISION**



The p.v. of the point Q, which divides externally the join of two given points A and B whose position vectors are $\mathbf{a}$ and $\mathbf{b}$ in the given ratio $m : n$, is

$$\mathbf{OQ} = \frac{m\mathbf{b} - m\mathbf{a}}{m-n}$$

SYSTEM OF VECTORS

An ordered set of three non-coplaner vectors is called system of vectors.

$(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are said to be forming right handed system if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] > 0$ and said to be forming left handed system if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] < 0$.

LINEAR COMBINATION

A vector $\mathbf{r}$ is said to be a linear combination of the given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$...etc., if there exist a system of scalars $x, y, z,$ ...etc. such that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + ...$$

LINEARLY DEPENDENT AND INDEPENDENT SYSTEM OF VECTORS

The system of n vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, ... x_n$ not all zero such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0} \qquad \text{… (1)}$$

The same system of vectors is said to be linearly independent, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + ... + x_n\mathbf{a}_n = \mathbf{0}$ implies that $x_1 = x_2 = ... = x_n = 0$ is the only solution.

COLLINEARITY OF THREE POINTS

The necessary and sufficient condition for three points with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ to be collinear is that there exist three scalars x, y, z not all zero such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \text{ where } x + y + z = 0.$$

TEST OF COLLINEARITY OF TWO VECTORS

To prove that two vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, find a scalar m such that one of the vectors is m times the other. In case no such scalar m exists, then the two vectors will be non-collinear vectors.

TEST OF COLLINEARITY OF THREE POINTS

Method I

To prove that three points A, B, C are collinear, find the vectors $\mathbf{AB}$ and $\mathbf{AC}$ and show that there exists a scalar m such that $\mathbf{AB} = m\mathbf{AC}$.

If no such scalar m exists, then the points are not collinear.

Method II

To prove that three points A, B, C with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively are collinear, find three scalars x, y, z (not all zero) such that

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}, \text{ where } x + y + z = 0.$$

If no such scalars x, y, z exist then the points are not collinear.

COPLANARITY OF FOUR POINTS

The necessary and sufficient condition for four points with position vectors **a, b, c** and **d** to be coplanar is that there exist scalars x, y, z and w not all zero such that

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+w\mathbf{d}=\mathbf{0}$$

where $$x+y+z+w=0.$$

TEST OF COPLANARITY OF THREE VECTORS

To prove that three vectors **a, b** and **c** to be coplanar, express one of these vectors as the linear combination of the other two i.e. write $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$.

Now, compare the coefficients from the two sides and find the values of x and y. If real values of scalars x and y exist then the vectors are coplanar otherwise non-coplanar.

TEST OF COPLANARITY OF FOUR POINTS

Method I

To prove that four points A, B, C and D are coplanar, find the vectors **AB, AC** and **AD** and then show that these three vectors are coplanar.

Method II

To prove that four points A, B, C and D with position vectors **a, b, c** and **d** respectively are coplanar, find four scalars x, y, z, w (not all zero) such that

$$x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+w\mathbf{d}=\mathbf{0} \text{ where } x+y+z+w=0.$$

If no such scalars x, y, z, w exist, then the points are non-coplanar.

Note :

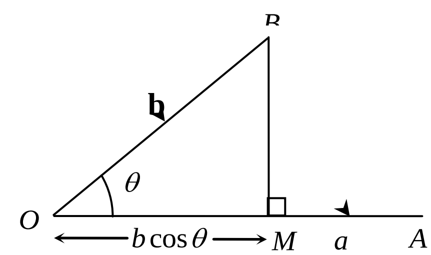
(i) In any linear vector equation if L.H.S. and R.H.S. contains two vectors then comparing the coefficient of vectors is allowed only when they are non-collinear.

(ii) In any linear vector equation if L.H.S. and R.H.S. contains three vectors then comparing coefficient of vectors is allowed only if the vectors are non-coplanar.

(iii) In any linear vector equation if L.H.S. and R.H.S. contains four or more than four vectors then comparing coefficients of vectors is never allowed.

SOME RESULTS ON LINEARLY DEPENDENT AND INDEPENDENT VECTORS

1. If **a, b, c** are non-coplanar vectors, then these are linearly independent and conversely if **a, b, c** are linearly independent, then they are non-coplanar.
2. If $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$, $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ and $\mathbf{c}=c_1\hat{\mathbf{i}}+c_2\hat{\mathbf{j}}+c_3\hat{\mathbf{k}}$ are three linearly dependent vectors, then
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$
3. Let **a, b, c** be three non-coplanar vectors. Then, vectors $x_1\mathbf{a}+y_1\mathbf{b}+z_1\mathbf{c}$, $x_2\mathbf{a}+y_2\mathbf{b}+z_2\mathbf{c}$ and $x_3\mathbf{a}+y_3\mathbf{b}+z_3\mathbf{c}$ will be coplanar if
$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}=0.$$
4. Any two non-collinear vectors are linearly independent.
5. Any two collinear vectors are linearly dependent.
6. Any three non-coplanar vectors are linearly independent.
7. Any three coplanar vectors are linearly dependent.
8. Any four vectors in 3-dimensional space are always linearly dependent.



SCALAR PRODUCT OF TWO VECTORS

The scalar product or dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the scalar $|\mathbf{a}||\mathbf{b}|\cos\theta,$ where θ is the angle between them such that $0 \le \theta \le \pi.$ It is denoted by $\mathbf{a}\cdot\mathbf{b}$ by placing a dot between the vectors.

Thus, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta.$

PROPERTIES OF SCALAR PRODUCT

1. Scalar product is commutative i.e. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ for any two vectors $\mathbf{a}$ and $\mathbf{b}$.
2. If m is any scalar and $\mathbf{a}$, $\mathbf{b}$ be any two vectors, then $(m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(m\mathbf{b})$
3. Scalar product is distributive w.r.t. vector addition i.e. for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$
$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}.$$
4. Magnitude of a vector as a scalar product : For any vector $\mathbf{a}$
$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = \mathrm{a}^2.$$
5. Scalar product of two perpendicular vectors is zero i.e. if $\mathbf{a}$ and $\mathbf{b}$ are two perpendicular vectors, then $\mathbf{a}\cdot\mathbf{b} = 0.$
However, if $\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow$ Either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} \perp \mathbf{b}$.
6. Scalar product of mutually orthogonal unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$:
$$\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$$
and $$\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = 0.$$
7. Scalar product of two vectors in terms of components : If
$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},$$
then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Thus, the scalar product of two vectors is equal to the sum of the product of their corresponding components.
8. Angle between two vectors in terms of the components of the given vectors.
If θ is the angle between two vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}},$ then
$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
9. Components of a vector **b** along and perpendicular to vector **a**
Component of $\mathbf{b}$ along $\mathbf{a}$ $= \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$
Component of $\mathbf{b}$ perpendicular to $\mathbf{a}$ $= \mathbf{b} - \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$
10. Any vector $\mathbf{r}$ can be expressed as $\mathbf{r} = (\mathbf{r}\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{r}\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{r}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}.$

SOME USEFUL IDENTITIES

Since scalar product satisfies commutative and distributive laws, we have

1. $(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}\cdot\mathbf{b}$
2. $(\mathbf{a}-\mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

3. $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}^2-\mathbf{b}^2.$

4. $(\mathbf{a}\cdot\mathbf{b})^2\le\mathbf{a}^2\mathbf{b}^2$ this relation is known as **Cauchy – Schwarz is inequality**

WORK DONE BY A FORCE

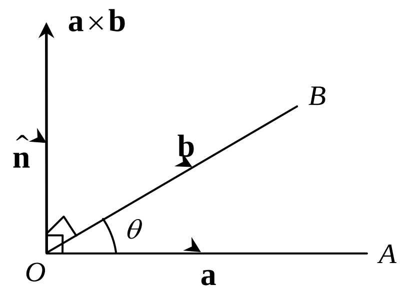
Work done by a force $\mathbf{F}$ in displacing a particle from A to B is defined by

$$W=\mathbf{F}\cdot\mathbf{AB}.$$

Note : If a number of forces are acting on a particle, then the sum of the works done by the separate forces is equal to the work done by the resultant force

CROSS PRODUCT OR VECTOR PRODUCT OF TWO VECTORS

The vector product or cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as a vector, written as $\mathbf{a}\times\mathbf{b}$, whose



(i) modulus is $|\mathbf{a}||\mathbf{b}|\sin\theta$, θ being the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$ and $0\le\theta\le\pi$.

(ii) direction is that of the unit vector $\hat{n}$ which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{n}$ form a right handed system.

Thus, $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\ \hat{\mathbf{n}}.$

Vector product in terms of components

Let $\mathbf{a}=a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}}$ and $\mathbf{b}=b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}}$ then $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Note :

1. By right handed system we mean that as the first vector $\mathbf{a}$ is turned towards the second vector $\mathbf{b}$ through an angle θ, $\hat{\mathbf{n}}$ will point in the direction in which a right handed screw would advance if turned in a similar manner.
2. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{O}$, we have $\mathbf{a}\times\mathbf{b}=\mathbf{O}$.
3. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$
4. A unit vector perpendicular to the plane of two given vectors $\mathbf{a}$ and $\mathbf{b}$ is $\hat{\mathbf{n}}=\dfrac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$.
5. $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$.

PROPERTIES OF VECTOR PRODUCT

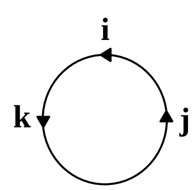
1. Vector product is not commutative i.e. $\mathbf{a}\times\mathbf{b}\ne\mathbf{b}\times\mathbf{a}$. In fact, $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.
2. Vector product is associative with respect to a scalar i.e. If m and n be any scalars and $\mathbf{a}$, $\mathbf{b}$ any vectors, then

$$m(\mathbf{a}\times\mathbf{b})=(m\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(m\mathbf{b})=(\mathbf{a}\times\mathbf{b})m;$$

$$(m\mathbf{a})\times(n\mathbf{b})=(n\mathbf{a})\times(m\mathbf{b})=(mn\mathbf{a})\times b$$

$$=\mathbf{a}\times(mn\mathbf{b})=mn(\mathbf{a}\times\mathbf{b}).$$

3. Vector product is distributive w.r.t. addition i.e. $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
4. If two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a}\times\mathbf{b}=\mathbf{0}$. In particular, $\mathbf{a}\times\mathbf{a}=\mathbf{0}$.
5. Vector product of mutually orthogonal unit vectors, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$:



$$\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=\mathbf{0}$$

and $\quad\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}=-\hat{\mathbf{j}}\times\hat{\mathbf{i}},\quad\hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}}=-\hat{\mathbf{k}}\times\hat{\mathbf{j}}$

$$= \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}}.$$

Note:

(i) $|\mathbf{a} \times \mathbf{b}|$ represent the scalar area of the parallelogram whose adjacent sides are **a** and **b**

(ii) If two sides of a triangle lie along the vector **a** and **b**, then $\frac{\mathbf{a}\times\mathbf{b}}{2}$ represent vector area of Δ.

(iii) If the diagonals of parallelogram lie along the vectors **a** and **b**, then $\frac{1}{2}(\mathbf{a}\times\mathbf{b})$ represent the vector **area** of parallelogram where as $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ represent its **scalar area.**

(iv) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the position vector of the vertices of the triangle ABC then **vector area** of the triangle is given by $= \frac{1}{2}[\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}]$ and $\frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$ be its **scalar area.**

Area should be zero if A, B, C are collinear.

(v) **Lagrange's Identity:** If a, b are any two vectors, then

$$|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \text{ or } |\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2$$

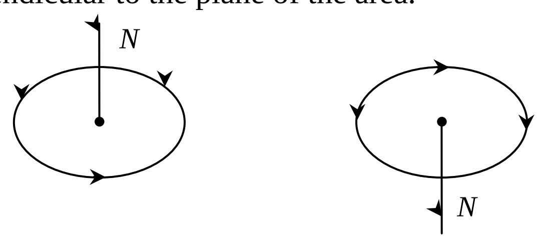
VECTOR AREA

Definition

It is possible to associate a direction to a plane area bounded by a closed curve which does not cross itself according as the direction of rotation by which the curve is drawn traced out is right handed or left handed. If the area is represented by a vector **A**, defined as follows :

(i) The magnitude of A is equal to the number of units of the given area.

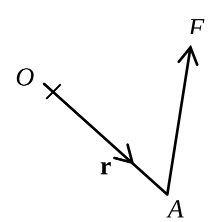
(ii) The support of **A**, is perpendicular to the plane of the area.



(iii) The sense of **A** is such that the direction of description of the boundary of the curve and the sense of **A** corresponded to the rotation of a right hand screw. The sense of A will be reversed, if we reverse the direction of description of the boundary of the area.

MOMENT OF A FORCE ABOUT A POINT

The vector moment or torque **m** of a force **F** acting at a point A about the point O is given by



$$\mathbf{M} = \mathbf{r}\times\mathbf{F} = \mathbf{OA}\times\mathbf{F}$$

where $\mathbf{r} = \mathbf{OA}$ is the p.v. of the point A w.r.t. the point O.

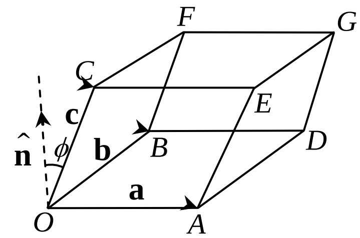
Note : The algebraic sum of the moments of a system of forces about any point is equal to the moment of their resultant about the same point.

TRIPLE PRODUCTS

SCALAR TRIPLE PRODUCT

If **a, b, c** be three vectors, the scalar product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ of the two vectors **a** and $\mathbf{b}\times\mathbf{c}$ is a scalar quantity, called the scalar triple product of these vectors **a, b** and **c** and is denoted by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

GEOMETRICAL INTERPRETATION OF SCALAR TRIPLE PRODUCT



Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three vectors. Consider a parallelopiped having coterminus edges $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ such that $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Then $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Let ϕ be the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$. If $\hat{\mathbf{n}}$ is a unit vector along $\mathbf{a} \times \mathbf{b}$, then ϕ is the angle between $\hat{\mathbf{n}}$ and $\mathbf{c}$.

Now, $\quad [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

PROPERTIES OF SCALAR TRIPLE PRODUCT

1. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \mathbf{b}.(\mathbf{c}\times\mathbf{a}) = \mathbf{c}.(\mathbf{a}\times\mathbf{b}).$$

   i.e. a cyclic permutation of three vectors does not change the value of the scalar triple product.

2. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = -\mathbf{b}.(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}.(\mathbf{b}\times\mathbf{a}) = -\mathbf{a}.(\mathbf{c}\times\mathbf{b})$$

   i.e. an anti-cyclic permutation of three vectors changes the value of the scalar triple product in sign but not in magnitude.

3. The positions of dot and cross can be interchanged without any change in the value of the scalar triple product i.e.

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b}).\mathbf{c}$$

4. $[\mathbf{i}\ \ \mathbf{j}\ \ \mathbf{k}] = 1$
5. The scalar triple product of three vectors is zero if any two of them are equal.
6. Scalar triple product vanishes if any two of its vectors are parallel or collinear.
7. The necessary and sufficient condition that the three non-zero, non-collinear vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar is their scalar triple product must vanish i.e. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.
8. $[\mathbf{a}+\mathbf{b}\ \ \mathbf{b}+\mathbf{c}\ \ \mathbf{c}+\mathbf{a}] = 2[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]$
9. $[\lambda\mathbf{a}\ \mu\mathbf{b}\ \nu\mathbf{c}] = \lambda\mu\nu[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]$

SCALAR TRIPLE PRODUCT IN TERMS OF COMPONENTS

1. If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\quad \mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$.

   Then, $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

2. If $\mathbf{a} = a_1\mathbf{p} + a_2\mathbf{q} + a_3\mathbf{r}$, $\quad \mathbf{b} = b_1\mathbf{p} + b_2\mathbf{q} + b_3\mathbf{r}$ and $\mathbf{c} = c_1\mathbf{p} + c_2\mathbf{q} + c_3\mathbf{r}$

   Then, $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{p},\ \mathbf{q},\ \mathbf{r}]$.

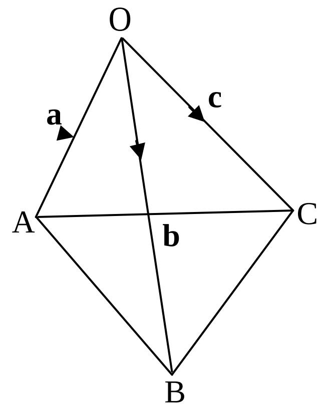
VOLUME OF A TETRAHEDRON

The volume of tetrahedron, whose three coterminous edges in the right –handed system are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by

$$= \frac{1}{3}(\text{Area of base}) \times \text{height}$$

$$= \frac{1}{3}\ (\text{Area of } \Delta\ ABC) \times \text{projection } \mathbf{OA} \text{ on } \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector $\perp r$ to the plane of $\Delta\,ABC$



$$= \frac{1}{3}\left[\frac{1}{2}|\mathbf{AB}\times\mathbf{AC}|\right]\frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{|\mathbf{AB}\times\mathbf{AC}|}$$

$$= \frac{1}{6}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$$

$$= \frac{1}{6} \text{ Volume of parallelopiped whose coterminous edges are } \mathbf{a}, \mathbf{b}, \mathbf{c}$$

$$= \frac{1}{6}[\mathbf{OA}, \mathbf{OB}, \mathbf{OC}]$$

VECTOR TRIPLE PRODUCT

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors, then the vectors $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ are called vector triple product of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$ i.e., (First . Third) Second – (First . Second) Third.

Note

1. Vector triple product is not associative

   i.e. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \neq \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

2. $$(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = -[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]$$
   $$= -[(\mathbf{a}\cdot\mathbf{c})\cdot\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}]$$
   $$= (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}.$$

3. $2\mathbf{r} = \hat{\mathbf{i}}\times(\mathbf{r}\times\hat{\mathbf{i}}) + \hat{\mathbf{j}}\times(\mathbf{r}\times\hat{\mathbf{j}}) + \hat{\mathbf{k}}\times(\mathbf{r}\times\hat{\mathbf{k}})$ and $2\mathbf{r} = (\hat{\mathbf{i}}\times\hat{\mathbf{r}})\times\hat{\mathbf{i}} + (\hat{\mathbf{j}}+\hat{\mathbf{r}})\times\hat{\mathbf{j}} + (\hat{\mathbf{k}}\times\hat{\mathbf{r}})\times\hat{\mathbf{k}}$

SCALAR PRODUCT OF FOUR VECTORS

If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors, then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is the scalar product of four vectors.

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

VECTOR PRODUCT OF FOUR VECTORS

If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors, then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is the vector product of four vectors.

EXPANSION OF VECTOR PRODUCT OF FOUR VECTORS

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$$
$$= [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}.$$

Note :

1. $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a} - [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} + [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d} = \mathbf{O}$.
2. Any vector $\mathbf{r}$ can be expressed in terms of three non-coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the form

$$\mathbf{r} = \frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\mathbf{a} + [\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\mathbf{b} + [\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}.$$

RECIPROCAL SYSTEM OF VECTORS

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors so that $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$, then the three vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ defined by the equations

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

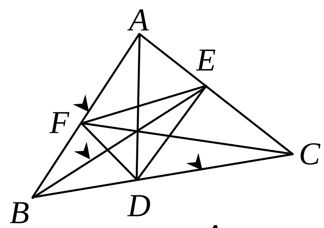
are called the reciprocal system of vectors to the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

PROPERTIES OF RECIPROCAL SYSTEM OF VECTORS

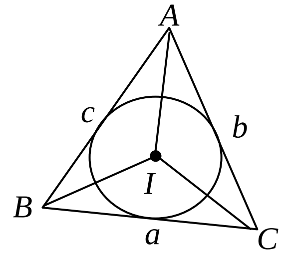
1. $\mathbf{a}\cdot\mathbf{a}' = \mathbf{b}\cdot\mathbf{b}' = \mathbf{c}\cdot\mathbf{c}' = 1$.
2. $\mathbf{a}\cdot\mathbf{b}' = \mathbf{a}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{c}' = \mathbf{b}\cdot\mathbf{a}' = \mathbf{c}\cdot\mathbf{a}' = \mathbf{c}\cdot\mathbf{b}' = 0$.
3. $\left[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'\right] = \dfrac{1}{\left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right]}$. This shows that both $\left[\mathbf{a}, \mathbf{b}, \mathbf{c}\right]$ and $\left[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}\right]$ are positive or both negative i.e., both form right handed system or both left handed system.
4. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar iff so are $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.
5. Orthonormal triad of vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ is self-reciprocal.
   i.e. $\hat{\mathbf{i}}' = \hat{\mathbf{i}},\ \hat{\mathbf{j}}' = \hat{\mathbf{j}},\ \hat{\mathbf{k}}' = \hat{\mathbf{k}}$.
6. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors for which $\left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right] \neq 0$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ constitute the reciprocal system of vectors, then any vector $\mathbf{r}$ can be expressed as
$$\mathbf{r} = (\mathbf{r}\cdot\mathbf{a}')\mathbf{a} + (\mathbf{r}\cdot\mathbf{b}')\mathbf{b} + (\mathbf{r}\cdot\mathbf{c}')\mathbf{c}.$$
$$= (\mathbf{r}\cdot\mathbf{a})\mathbf{a}' + (\mathbf{r}\cdot\mathbf{b})\mathbf{b}' + (\mathbf{r}\cdot\mathbf{c})\mathbf{c}'.$$

SOME IMPORTANT FACTS

1. (i) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}| \Leftrightarrow \mathbf{a}\perp\mathbf{b}$
   (ii) $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}|+|\mathbf{b}| \Leftrightarrow \mathbf{a}\parallel\mathbf{b}$ i.e. $\mathbf{a} = \lambda\mathbf{b}$
   (iii) $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2 \Leftrightarrow \mathbf{a},\mathbf{b}$ are orthogonal i.e. $\mathbf{a}\perp\mathbf{b}$
2. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are mutually perpendicular i.e. $\mathbf{a}\perp\mathbf{b}\perp\mathbf{c}$ and equal in magnitudes, then $(\mathbf{a}+\mathbf{b}+\mathbf{c})$ is equally inclined with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and angle of inclination with every one is given by $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$
3. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors then
   (i) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) + \mathbf{b}\times(\mathbf{c}+\mathbf{a}) + \mathbf{c}\times(\mathbf{a}+\mathbf{b}) = \mathbf{0}$
   (ii) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$
4. There vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar if and if only.
   (i) $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, $\mathbf{c}+\mathbf{a}$ are coplanar.
   (ii) $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ are coplanar.
5. If *D, E, F* are the midpoints of the sides *BC, CA, AB* of a triangle then $\mathbf{AD}+\mathbf{BE}+\mathbf{CF} = \mathbf{0}$



6. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the position vector of an equilateral triangle whose orthocentre is at origin then $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$ as in equilateral triangle circumcentre, orthocentre, incentre and centroid coincide.
7. If I is the centre of the circle inscribed in a triangle ABC then the value of
$$a\,\overrightarrow{IA} + b\,\overrightarrow{IB} + c\,\overrightarrow{IC} = \mathbf{0}$$
   where $BC = a$, $CA = b$, $AB = c$.
$$I = \frac{\mathrm{a}\mathbf{a}+\mathrm{b}\mathbf{b}+\mathrm{c}\mathbf{c}}{\mathrm{a}+\mathrm{b}+\mathrm{c}} = \frac{|\mathbf{b}-\mathbf{c}|\mathbf{a}+|\mathbf{c}-\mathbf{a}|b+|\mathbf{a}-\mathbf{b}|\mathbf{c}}{|\mathbf{b}-\mathbf{c}|+|\mathbf{c}-\mathbf{a}|+|\mathbf{a}-\mathbf{b}|}$$



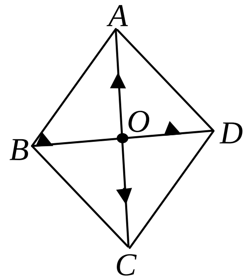
8. **Orthocenter formula**

The position vector of the orthocenter of ΔABC is

$$\frac{\mathbf{a}\tan A+\mathbf{b}\tan B+\mathbf{c}\tan C}{\tan A+\tan B+\tan C}.$$

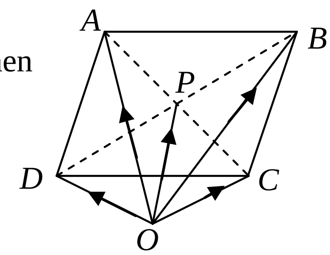
9. If $ABCD$ is a rhombus whose diagonals at the origin then

$$\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}=\mathbf{0}$$



10. If $ABCD$ is a parallelogram whose diagonals intersect at P if O be the origin then

$$\mathbf{OA}+\mathbf{OB}+\mathbf{OC}+\mathbf{OD}=4\mathbf{OP}$$



11. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are position vector of three points respectively then $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$ is perpendicular to the plane of ABC.

12. (a) $[\mathbf{a}+\mathbf{b},\ \mathbf{b}+\mathbf{c},\ \mathbf{c}+\mathbf{a}]=2[\mathbf{a},\ \mathbf{b},\ \mathbf{c}]$

(b) $[\mathbf{a}-\mathbf{b},\ \mathbf{b}-\mathbf{c},\ \mathbf{c}-\mathbf{a}]=0$

(c) $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{a},\ \mathbf{b},\ \mathbf{c}]^2$

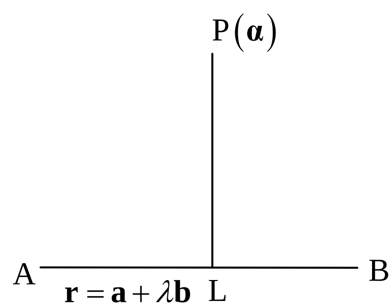
(d) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]=\begin{vmatrix}\mathbf{a}\cdot\mathbf{p} & \mathbf{a}\cdot\mathbf{q} & \mathbf{a}\cdot\mathbf{r}\\ \mathbf{b}\cdot\mathbf{p} & \mathbf{b}\cdot\mathbf{q} & \mathbf{b}\cdot\mathbf{r}\\ \mathbf{c}\cdot\mathbf{p} & \mathbf{c}\cdot\mathbf{q} & \mathbf{c}\cdot\mathbf{r}\end{vmatrix}$

(e) $\mathbf{r}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]\mathbf{a}+[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]\mathbf{b}+[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]\mathbf{c}$

(f) $\mathbf{r}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=(\mathbf{r}.\mathbf{a})(\mathbf{b}\times\mathbf{c})+(\mathbf{r}.\mathbf{b})(\mathbf{c}\times\mathbf{a})+(\mathbf{r}.\mathbf{c})(\mathbf{a}\times\mathbf{b})$

(g) $(\mathbf{a}\times\mathbf{b})[\mathbf{l}\ \mathbf{m}\ \mathbf{n}]=\begin{vmatrix}\mathbf{l} & \mathbf{m} & \mathbf{n}\\ \mathbf{a}.\mathbf{l} & \mathbf{a}.\mathbf{m} & \mathbf{a}.\mathbf{n}\\ \mathbf{b}.\mathbf{l} & \mathbf{b}.\mathbf{m} & \mathbf{b}.\mathbf{n}\end{vmatrix}$

13. **Perpendicular distance of a point from a line**



Let L is the foot of perpendicular drawn from $P(\boldsymbol{\alpha})$ on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. Since $\mathbf{r}$ denotes the position vector of any point on the line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$.

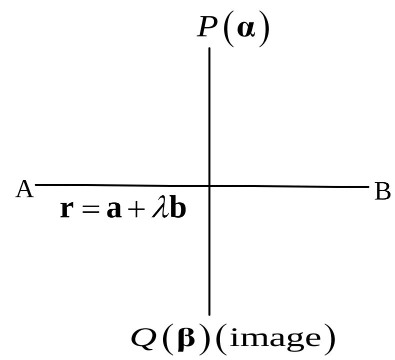
So, let the position vector of L be $\mathbf{a}+\lambda\mathbf{b}$.

Position vector of $\text{L}=\mathbf{a}-\left(\frac{(\mathbf{a}-\boldsymbol{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

and $\mathbf{PL}=(\mathbf{a}-\boldsymbol{\alpha})-\left(\frac{(\mathbf{a}-\boldsymbol{\alpha}).\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$

The length PL, is the magnitude of $\mathbf{PL}$, and is the required length of perpendicular.

14. **Image of a point in a straight line**



Let $Q(\boldsymbol{\beta})$ is the image of P in $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, then

p.v. of Q is $\boldsymbol{\beta}=2\mathbf{a}-\boldsymbol{\alpha}-\left(\dfrac{2\left(\mathbf{a}-\boldsymbol{\alpha}\right)\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$ and

$\mathbf{PQ}=2\mathbf{a}-2\boldsymbol{\alpha}-\left(\dfrac{2\left(\mathbf{a}-\boldsymbol{\alpha}\right)\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right).$

15. **Shortest distance between two parallel lines:** Let l_1 and l_2 be two lines whose equations are $l_1:\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $l_2:\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$ respectively.
Then, shortest distance between them

$$=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|=\left|\frac{[\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{a}_2-\mathbf{a}_1]}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$$

Shortest distance between two parallel lines

The shortest distance between the parallel lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}$ an d $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}$

is given by $d=\dfrac{|(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}|}{|\mathbf{b}|}.$

16. Vector equation of a plane through the point $A(\mathbf{a})$ and perpendicular to the vector $\mathbf{n}$ is $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n}=0$ or $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$ or $\mathbf{r}\,.\,\mathbf{n}=$ d, where d $=\mathbf{a}\,.\,\mathbf{n}$. This is known as the *scalar product* form of a plane.

17. Vector equation of a plane normal to unit vector $\hat{\mathbf{n}}$ and at a distance d from the origin is $\mathbf{r}.\hat{\mathbf{n}}=d$. If ***n*** is not a unit vector, then to reduce the equation $\mathbf{r}\,.\,\mathbf{n}=d$ to normal form we divide both sides by $|\mathbf{n}|$ to obtain $\mathbf{r}\cdot\dfrac{\mathbf{n}}{|\mathbf{n}|}=\dfrac{d}{|\mathbf{n}|}$ or $\mathbf{r}\cdot\hat{\mathbf{n}}=\dfrac{d}{|\mathbf{n}|}$

18. The equation of the plane passing through a point having position vector **a** and parallel to **b** and **c** is
$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$ or $[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ where λ and μ are scalars.

19. Vector equation of a plane passing through a point $\mathbf{a},\mathbf{b},\mathbf{c}$ is
$\mathbf{r}=(1-s-t)\mathbf{a}+s\mathbf{b}+t\mathbf{c}$ or $\mathbf{r}\cdot(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})=[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}].$

20. The equation of any plane through the intersection of planes
$\mathbf{r}\cdot\mathbf{n}_1=d_1$ and $\mathbf{r}\cdot\mathbf{n}_2=d_2$ is $\mathbf{r}\cdot(\mathbf{n}_1+\lambda\mathbf{n}_2)=d_1+\lambda d_2$ where λ is an arbitrary constant.

21. The perpendicular distance of a point having position vector **a** from the plane $\mathbf{r}\cdot\mathbf{n}=d$ is given by
$p=\dfrac{|\mathbf{a}\cdot\mathbf{n}-d|}{|\mathbf{n}|}.$

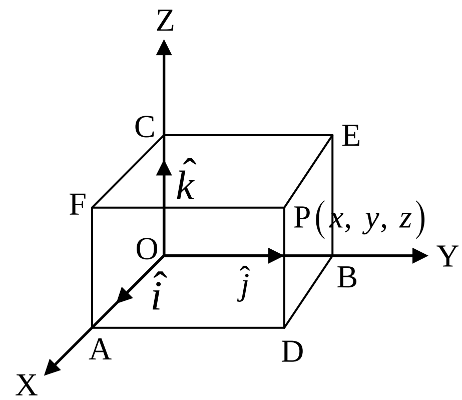
22. Perpendicular distance of a point P($\mathbf{r}$) from a plane passing through the points **a**, **b** and **c** is given by
$PM=\dfrac{(\mathbf{r}-\mathbf{a}).(\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})}{|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}|}.$

23. Angle between line and plane: If θ is the angle between a line $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and the plane $\mathbf{r}\cdot\mathbf{n}=d$, then
$\sin\theta=\dfrac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}.$

24. The equation of sphere with centre at C ($\mathbf{c}$) and radius '*a*' is $|\mathbf{r}-\mathbf{c}|=a.$

3-Dimensional Coordinate Geometry — Chapter 32

1. **Co-ordinates of a point in space**

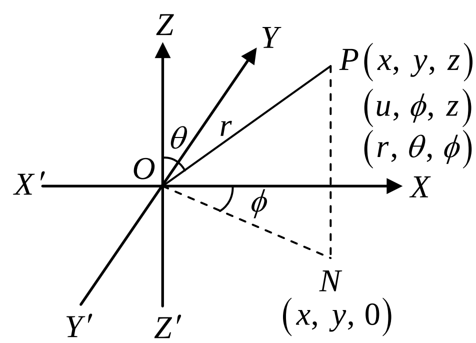


Cartesian co-ordinates : Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as *x*-axis, *y*-axis and *z*-axis respectively, in such a way that they form a right-handed system.

The plane XOY, YOZ and ZOX are known as *xy*-plane, *yz*-plane and *zx*-plane respectively. Also, $OA = x$, $OB = y$, $OC = z$.

The three co-ordinate planes $(XOY, YOZ \text{ and } ZOX)$ divide space into eight parts and these parts are called octants.

Sign of co-ordinates of a point : The signs of the co-ordinates of a point in three dimension follow the convention that all distance measured along or parallel to OX, OY, OZ will be positive and distance moved along a parallel to OX', OY', OZ' will be negative.



Cylindrical co-ordinates : If the rectangular cartesian co-ordinates of P are (x, y, z), then those of N are $(x, y, 0)$ and we can easily have the following relations :

$$x = u\cos\phi,\ y = u\sin\phi \text{ and } z = z.$$

Hence, $u^2 = x^2 + y^2$ and $\phi = \tan^{-1}(y/x)$.

Cylindrical co-ordinates of $P \equiv (u, \phi, z)$.

Spherical polar co-ordinates : The measures of quantities r, θ, ϕ are known as spherical or three dimensional polar co-ordinates of the point P. If the rectangular cartesian co-ordinates of P are (x, y, z) then $z = r\cos\theta$, $u = r\sin\theta$.

$$\therefore \quad x = u\cos\phi = r\sin\theta\cos\phi, \quad y = u\sin\phi = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta.$$

Also, $r^2 = x^2 + y^2 + z^2$ and $\tan\theta = \dfrac{u}{z} = \dfrac{\sqrt{x^2+y^2}}{z}$; $\tan\phi = \dfrac{y}{x}$.

2. **Distance Formula :** Distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3. **Section Formula**

(i) Co-ordinates of the point which divides the join of P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) **internally** in the ratio $m : n$ are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$.

(ii) Co-ordinates of the point which divides the join of P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) **externally** in the ratio $m : n$ are $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}, \dfrac{mz_2 - nz_1}{m-n}\right)$.

4. **Triangle and tetrahedron**

(i) **Co- ordinates of the centroid**

(a) If $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_3, y_3 z_3)$ are the vertices of a triangle, then co-ordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

(b) If $(x_r, y_r, z_r); r = 1, 2, 3, 4$ are vertices of a tetrahedron then co-ordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

(iii) **Area of triangle:** Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of a triangle, then

$$\Delta_x = \frac{1}{2}\left|\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}\right|, \quad \Delta_y = \frac{1}{2}\left|\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}\right| \quad \Delta_z = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

Now, area of ΔABC is given by the relation $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

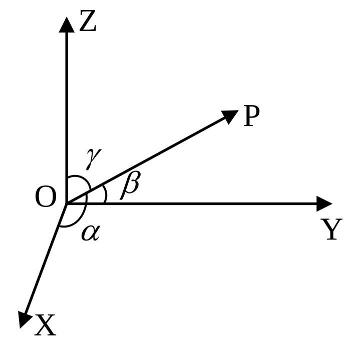
(iv) **Condition of collinearity:** Points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are collinear,

if $\dfrac{x_1 - x_2}{x_2 - x_3} = \dfrac{y_1 - y_2}{y_2 - y_3} = \dfrac{z_1 - z_2}{z_2 - z_3}$.

(v) **Volume of tetrahedron:** If vertices of tetrahedron be

$$(x_r, y_r, z_r); r = 1, 2, 3, 4; \text{ then } V = \frac{1}{6}\left|\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\right|$$

5. **Direction Cosines**



Let a vector $\overrightarrow{OP}$ (or line OP) makes angles α, β and γ with positive *X, Y, Z* axis respectively.

Then $\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of vector $\overrightarrow{OP}$ (or line OP) and these are denoted by *l, m, n* respectively.

Note :

(a) Direction cosines of X, Y and Z axis are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

(b) If P(x, y, z) is point in space, $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then projection of $\vec{r}$ on X, Y, Z axis are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$ respectively.

(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ *i.e.,* $l^2 + m^2 + n^2 = 1$.

(d) $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = l\hat{i} + m\hat{j} + n\hat{k}$.

6. **Direction Ratios**

A set of three numbers *a, b, c* (not all zero) is said to be direction ratios of a line OP or $\overrightarrow{OP}$ *if* $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$, where l, m, n are direction cosines.

Note :

(a) If *a, b, c* are direction-ratios of a line , then it's direction cosines are

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}; \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}; \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

where + ve or – ve sign is to be taken in all the three.

(b) A line can have many sets of direction ratio which are proportional to each other.

(c) If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two points, then direction-ratios of $\overrightarrow{PQ}$ (or line) PQ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

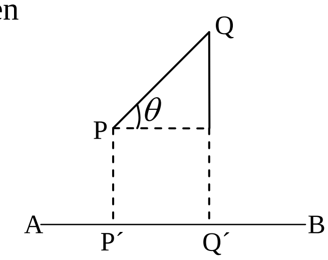
(d) Angle between two vectors (or line) whose direction-ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2) is given by

$$\therefore \qquad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(i) If these are '$\perp$'; $a_1a_2 + b_1b_2 + c_1c_2 = 0.$ (ii) If these are '$\|$'; $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

7. **Projection**

Projection of a line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line AB whose direction cosines are *l*, *m* and *n*. If the line segment *PQ* makes angle θ with the line *AB* then



Projection of PQ is P'Q', $\therefore$ P'Q' $= \left| (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma \right|$

$$= \left| (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n \right|.$$

PLANE

1. (i) **General equation**

If *a, b, c* are direction-ratios of normal to the plane, then the equation of plane in Cartesian form is $ax + by + cz + d = 0$

In vector form general equation is

$\vec{r}.\vec{n} = p$; where $\vec{n}$ is a vector $\perp$ to the plane.

(ii) **Equation of plane in normal form**

If *l, m, n* be the direction cosines of the normal to a plane and *p* be the length of the perpendicular from the origin on the plane, then the equation of the plane is $lx + my + nz = p$.

In vector form normal equation of plane is

$\vec{r} \cdot \hat{n} = p$ where $\hat{n}$ is a unit vector $\perp$ to the plane.

(iii) **Equation of plane in intercepts form**

If a plane makes intercepts *a, b, c* on the axes of coordinates, its equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

(iv) **Equation of a plane through a given point**

$P(x_1, y_1, z_1)$ and having (a, b, c) as direction-ratios of normal is given by :

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ *Cartesian form*

Or

$\left(\vec{r}-\vec{a}\right)\cdot\vec{n}=0;\quad \vec{n}$ is a vector normal to plane. *Vector form*

where $\vec{a}$ is the position vector of a point in the plane.

(v) **Equation of a plane through three given points**

The equation of the plane passing through three non-collinear points $\left(x_1,\, y_1,\, z_1\right), \left(z_2,\, y_2,\, z_2\right)$ and $\left(x_3,\, y_3,\, z_3\right)$ is

$$\begin{vmatrix} \left(x-x_1\right) & \left(y-y_1\right) & \left(z-z_1\right) \\ \left(x_2-x_1\right) & \left(y_2-y_1\right) & \left(z_2-z_1\right) \\ \left(x_3-x_1\right) & \left(y_3-y_1\right) & \left(z_3-z_1\right) \end{vmatrix}=0.$$

2. **Equation of Systems of Planes**

(i) The equation $ax+by+cz+k=0$ represents a system of plane parallel to the plane $ax+by+cz+d=0$, k being parameter.

Or

Vector form $\vec{r}.\vec{n}=k$

(ii) The equation $ax+by+cz+k=0$, represents a system of planes perpendicular to the line $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

(iii) The equation $\left(a_1x+b_1y+c_1z+d_1\right)+k\left(a_2x+b_2y+c_2z+d_2\right)=0$ .represents a system of planes passing through the intersection of the planes $a_1x+b_1y+c_1z+d_1=0,$ and $a_2x+b_2y+c_2z+d_2=0.$ k being a parameter.

(iv) The equation $A\left(x-x_1\right)+B\left(y-y_1\right)+C\left(z-z_1\right)=0$ represents a system of planes passing through the point $\left(x_1,y_1,z_1\right)$ where A, B, C are parameters.

3. **Angle Between Two Planes**

The angle θ between the planes $a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0,$ is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$$

4. **Parallelism and Perpendicularity of Two Planes**

The planes $a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0$ are parallel if and only if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$; and perpendicular if and only if $a_1a_2+b_1b_2+c_1c_2=0$

5. **Two Sides of a Plane**

Two points $P\left(x_1,y_1,z_1\right)$ and $Q\left(x_2,y_2,z_2\right)$ lie on the same or different sides of the plane $ax+by+cz+d=0$ according as $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the same or of different signs.

6. **Length of the Perpendicular from a Point to a Plane**

The perpendicular distance of the point $\left(x_1,y_1,z_1\right)$ from the plane $lx+my+nz=p$ is $\left|\,p-lx_1-my_1-nz_1\,\right|$, where l, m, n are direction cosines of the normal to the plane and p is the length of the perpendicular on the plane from the origin. The perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{(a^2+b^2+c^2)}}$

7. **Distance between two Parallel Planes**

$ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

8. **Bisectors of the Angles between Two Planes**

Let $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be the equations of two planes, written in such a way that d_1 and d_2 are both positive. Then

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

is the equation of the plane bisecting that angle between the given planes which contains the origin, and

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

is the equation of the plane bisecting that angle between the given planes which does not contain the origin.

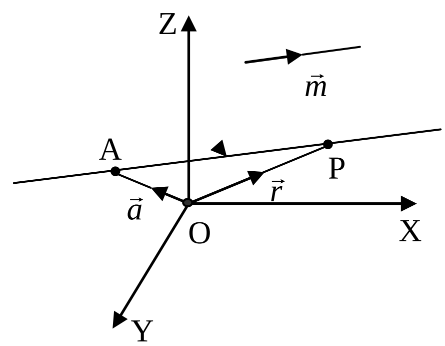
(i) If angle between bisector plane and one of the plane is less than 45^o, then it is acute angle bisector, otherwise it is obtuse angle bisector.

(ii) If $a_1a_2+b_1b_2+c_1c_2$ is negative, then origin lies in the acute angle between the given planes provided d_1 and d_2 are of same sign and if $a_1a_2+b_1b_2+c_1c_2$ is positive, then origin lies in the obtuse angle between the given planes.

STRAIGHT LINE

1. (i) **Vector equation** of a line passing through a point 'A' whose position vector is $\vec{a}$ w.r.t. some origin O and is parallel to a given vector $\vec{m}$ is given by :

$\vec{r}=\vec{a}+\lambda\vec{m}$; λ is a constant.



(ii) **Cartesian equation**

If a straight line passes through a given point (x_1, y_1, z_1) and has direction cosines *l*, *m*, *n*, then the coordinates of any point on it satisfy the equations

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}(=r)$$

These equations enable us to write down the coordinates of any point (*x, y, z*) on the line in terms of its distance *r* is

$x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr.$ This form is called the ***distance form*** of the equation of a line.

A line passing through a point A(x_1, y_1, z_1) and having direction-ratios *a, b, c* is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda,\ \text{say.}$$

It is called **symmetric form** of the line.

Note : Any point on this line is $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where λ is a parameter.

2. **Equations of line through two given points**

Vector form. Let the points on line are A(x_1, y_1, z_1) and B(x_2, y_2, z_2) and their position vectors are $\vec{a}_1$ and $\vec{a}_2$ respectively. Then equation of line is given by $\vec{r} = \vec{a}_1 + \lambda(\vec{a}_2 - \vec{a}_1)$.

Cartesian form $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

The coordinates of a variable point on *AB* can be expressed in terms of a parameter λ in the form

$$x = \frac{\lambda x_2 + x_1}{\lambda + 1}, \quad y = \frac{\lambda y_2 + y_1}{\lambda + 1}, \quad z = \frac{\lambda z_2 + z_1}{\lambda + 1},$$

λ being any real number different from -1. In fact, (*x, y, z*) are the coordinates of the point which divides the join of *A* and *B* in the ratio $\lambda:1$.

3. **Changing unsymmetrical form to symmetrical form**

The unsymmetrical form of a line

$$ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0$$

can be changed to symmetrical form as follows :

$$\frac{x - \dfrac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \frac{y - \dfrac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

4. **Angle between two lines**

Vector form : Let the lines be $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$, then $\cos\theta = \dfrac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1|.|\vec{b}_2|}$.

If lines are (i) parallel $\vec{b}_1 = k\vec{b}_2$ and (ii) perpendicular $\vec{b}_1.\vec{b}_2 = 0$.

Cartesian form : Let the lines are $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ *and* $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

If these are (i) parallel $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (ii) perpendicular $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

5. **To find equation of a line parallel to given line**

Vector form : Given equation is $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$.

Equation of a line parallel to it through point B$(\vec{a}_2)$ is $\vec{r} = \vec{a}_2 + \mu\vec{b}_1$.

Cartesian form : Let the given line be $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ then a line parallel to it and passing through point B(x_2, y_2, z_2) has equation $\dfrac{x - x_2}{a} = \dfrac{y - y_2}{b} = \dfrac{z - z_2}{c}$.

6. **To find equation of line perpendicular to two given lines**

Vector form

Let given lines be: $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$.

A line perpendicular to them is $\vec{r} = \vec{a}_3 + \alpha\vec{b}_3$ where $\vec{b}_1.\vec{b}_3 = 0$ and $\vec{b}_2.\vec{b}_3 = 0$ or $\vec{b}_3 = \vec{b}_1 \times \vec{b}_2$.

Cartesian form

Let given lines be : $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$

Then a line through (x_3, y_3, z_3) and $\perp$ to given lines is: $\dfrac{x-x_3}{a}=\dfrac{y-y_3}{b}=\dfrac{z-z_3}{c}$

where $aa_1+bb_1+cc_1=0$ and $aa_2+bb_2+cc_2=0$.

Find *a*, *b*, *c* by rule of cross multiplication.

7. **To find whether two given lines cut or not**

Vector form. Let given lines be $\vec{r}_1=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}_2=\vec{a}_2+\mu\vec{b}_2$.

Since $\vec{r}$ is the position vector of an arbitrary point on the lines, if these intersect, for some values of λ and μ these points must coincide *i.e.*, $\vec{a}_1+\lambda\vec{b}_1=\vec{a}_2+\mu\vec{b}_2$. Equate components of $\hat{i}, \hat{j}, \hat{k}$,

Solve any two equations for 'λ' and 'μ'. Put this value in third equation, if it is satisfied lines intersect, otherwise not. To find point of intersection put λ (or μ) in given equation.

Let given lines be $L_1:\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}=\lambda$, say and $L_2:\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}=\mu$, say.

Any point on line L_1 is $(x_1+\lambda a_1, y_1+\lambda b_1, z_1+\mu c_1)$ and on line L_2 is $(x_2+\mu a_2, y_2+\mu b_2, z_2+\mu c_2)$.

If lines intersect these two points must coincide for some value of λ and μ. Equate the corresponding co-ordinates then proceed as in vector form.

8. **To find foot of perpendicular from a point on a line**

Vector form

Let given point be $A(\vec{a}_1)$ and line is $\vec{r}=\vec{a}+\lambda\vec{b}$.

Position vector of any point on line is $P(\vec{a}+\lambda\vec{b})$.

Then $\overrightarrow{AP}=(\text{Position vector of P})-(\text{Position vector of A})$.

If P is foot of perpendicular, then $\overrightarrow{\text{AP}}.b=0$ find λ.

Put the value of λ in $\vec{a}+\lambda\vec{b}$ to get foot of '$\perp$'. Length of $\perp = \left|\overline{\text{AP}}\right|$

Equation of '$\perp$' is $\vec{r}=\vec{a}_1+\mu\left(\overline{\text{AP}}\right)$.

Cartesian form

Let $A(x_2, y_2, z_2)$ be given point and $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$ be the line.

Let $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=\lambda$, say.

Any point on this line is $P(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$.

Direction-ratios of $AP=((x_1+a\lambda)-x_2, (y_1+b\lambda)-y_2, (z_1+c\lambda)-z_2)$

If P is foot of $\perp$ form A on line, then use '$a_1a_2+b_1b_2+c_1c_2=0$' where (a_1, b_1, c_1) and (a_2, b_2, c_2) are direction-ratios of given line and AP respectively.

Find λ and put in P to get foot of $\perp$.

Length of perpendicular = AP and equation of perpendicular is : $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$.

9. **Line of shortest distance:**

If l_1 and l_2 are two skew lines, then the straight line which is perpendicular to each of these two non-intersecting lines is called the "Line of shortest distance "There is one and only one line perpendicular to each of lines l_1 and l_2."

10. **Shortest distance between two skew lines**

Vector form

Let two lines be $\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$ then Shortest Distance $= \left|\dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$.

Cartesian form

Let two skew lines be, $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$

Therefore, the shortest distance between the lines is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - l_1n_2)^2 + (l_1m_2 - m_1l_2)^2}}$$

Method of parallel plane

When one of the lines is in the general form and the other is in the symmetrical form

Let $\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\}$ … (1)

and $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ … (2)

are two equations of straight lines

Use the following steps to find distance between these lines

Step I : Find the equation of a plane which contain line (1)

i.e. $a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$ … (*)

Step II : Find 'λ' so that the plane (*) is parallel to the line (2)

Step III : Find the distance of the point $(x_1,\ y_1,\ z_1)$ from the plane (*)

This is the distance between these two lines.

Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to a given plane ax + by + cz + d = 0.

If AP be the perpendicular from A to the given plane, then it is parallel to the normal to the plane, so that its equation is

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = \frac{z - \gamma}{c} = r,\ (say)$$

Any point P on it is $(ar + \alpha, br + \beta, cr + \gamma)$. If it lies on the given plane and we find the value of r and hence the point P.

11. **Image of a point in a plane**

Let P and Q be two points and let L be a plane such that

(i) Line PQ is perpendicular to the plane L, and

(ii) Mid-point of PQ lies on the plane L.

Then either of the point is the image of the other in the plane L.

12. **To find the image of a point in a given plane, we proceed as follows**

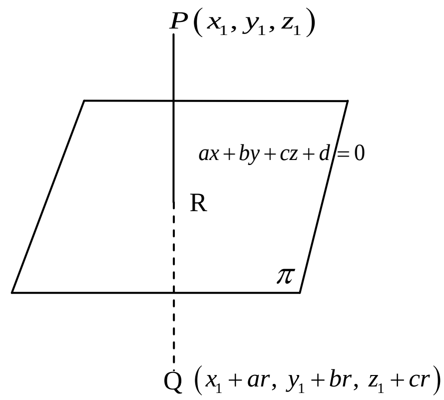
(i) Write the equations of the line passing through P and normal to the given plane as $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$

Write the co-ordinates of image Q as (x_1+ar, y_1+br, z_1+cr).

(ii) Find the co-ordinates of the mid-point R of PQ.

(iii) Obtain the value of r by putting the co-ordinates of R in the equation of the plane.

(iv) Put the value of r in the co- ordinates of Q.



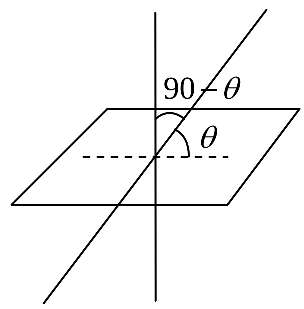
13. **Angle between a line and a plane**

Let $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$ be a line and $a_1x+b_1y+c_1z+d_1=0$ be a plane and θ the angle between them

then $$\cos(90-\theta)=\sin\theta=\frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^{\,2}+b_1^{\,2}+c_1^{\,2}}}$$

Or

If plane is $\vec{r}.\vec{n}=q$ and line is $\vec{r}=\vec{a}+\lambda\vec{m}$ then $\sin\theta=\dfrac{\vec{n}.\vec{m}}{|\vec{n}|.|\vec{m}|}$.



Note :

(a) Plane and straight line will be parallel if $a\,a_1+b\,b_1+c\,c_1=0$

(b) Plane and straight line will be perpendicular if $\dfrac{a}{a_1}=\dfrac{b}{b_1}=\dfrac{c}{c_1}$.

(c) The line may lie in the plane if $a\,a_1+b\,b_1+c\,c_1=0$ and $a_1x_1+b_1y_1+c_1z_1=0$.

SPHERE

1. Equation of a sphere having $C(\vec{c})$ as it's centre and a as it's radius:

$$|\vec{r}-\vec{c}|=a \quad \text{(Vector form)}$$

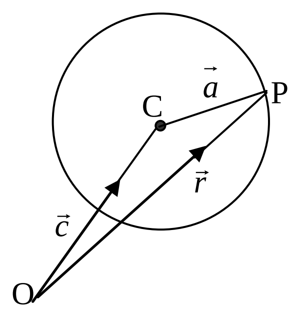
Cartesian form: Let center be $C(x_1, y_1, z_1)$ and radius = a.

Then equation is: $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2$. …(1)

If the centre is at the origin, then equation (1) takes the form

$$x^2+y^2+z^2=a^2$$

which is known as the standard form of the equation of the sphere.



2. **General equation of sphere**

The general equation of a sphere is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ with centre $(-u,-v,-w)$ i.e., $\left(-\tfrac{1}{2}\right.$ coefficient of x $,-\left(\tfrac{1}{2}\right)$ coefficient of y), $-\left(\tfrac{1}{2}\right)$(coefficient z) and, radius = $\sqrt{u^2+v^2+w^2-d}$.

Equation in sphere in various forms

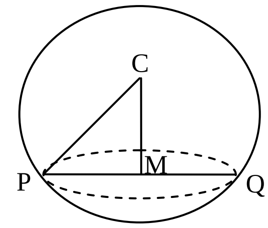
3. **Diameter form of the equation of a sphere:** If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the extremities of a diameter of a sphere, then its equation is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$

4. **Section of a sphere by a plane**

Consider a sphere intersected by a plane. The set of points common to both sphere is always a circle. The equation of the sphere and the plane taken together represent the plane section.

Let C be the centre of the sphere and M be the foot of the perpendicular from C on the plane. Then M is the centre of the circle and radius of the circle is given by $PM=\sqrt{CP^2-CM^2}$.



The centre M of the circle is the point of intersection of the plane and line CM which passes through C and is perpendicular to the given plane.

6. **Great circle:** The section of a sphere by a plane through the centre of the sphere is a great circle. Its centre and radius are the same as those of the given sphere.

7. **Condition of tangency of a plane to a sphere**

A plane touches a given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere. The plane $lx+my+nz=p$ touches the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ If $(ul+vm+wn-p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$.

8. **Intersection of straight line and a sphere**

Let the equation of the sphere and the straight line be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(1)$$

and $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r,(say) \qquad ...(2)$$

Any point of on the line (2) is $(\alpha+lr,\beta+mr,\gamma+nr)$. If this point lies on the sphere (1) then we have,

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0 \qquad \text{or}$$

$$r^2\left[l^2+m^2+n^2\right]+2r\left[l(u+\alpha)+m(v+\beta)+n(w+\gamma)\right]+\left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right)=0 \quad ...(3)$$

This is a quadratic equation in r and so gives two values of r and therefore the line (2) meets the sphere (1) in two points which may be real, coincident and imaginary, according as root of (3) are so.

If l,m,n are the actual direction cosines of the line, then $l^2+m^2+n^2=1$ and then the equation (3) can be simplified.

9. **Angle of intersection of two spheres**

If the angle of intersection of two spheres is a right angle, the sphere are said to be orthogonal.

Condition for orthogonality of two spheres:

Let the equation of the two sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ …(1)

and $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$ …(2)

If the sphere (1) and (2) cut orthogonally, then $2uu'+2vv'+2ww'=d+d'$, which is the require condition.

Note : Two sphere of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.

Probability

Chapter 33

SOME BASIC DEFINITIONS

EXPERIMENT

An operation which results in some well defined outcome is called an experiment.

RANDOM EXPERIMENT

An experiment whose outcome cannot be predicted with certainty is called a random experiment.

For example, tossing of a fair coin or throwing an unbiased die or drawing a card from a well shuffled pack of 52 cards is a random experiment.

SAMPLE SPACE

The set of all possible outcomes of a random experiment is called the sample space.

It is usually denoted by S.

For example, if we toss a coin, there are two possible outcomes, a head (H) or a tail (T).

So, the sample space in this experiment is given by $S = \{H, T\}$.

EVENT

A subset of the sample space S is called an Event.

Note :

(a) Sample space S plays the same role as the universal set for all problems related to the particular experiment.

(b) ϕ is also a subset of S which is called an **impossible event.**

(c) S is also a subset of S which called a **sure event.**

SIMPLE EVENT

An event having only a single sample point is called a simple event.

For example, when a coin is tossed, the sample space $S = \{H, T\}$.

Let $E_1 = \{H\} =$ the event of occurrence of head and

$E_2 = \{T\} =$ the event of occurrence of tail.

Then, E_1 and E_2 are simple events.

MIXED EVENT

A subset of the sample space S which contains more than one element is called a mixed event.

For example, when a coin is tossed, the sample space $S = \{H, T\}$.

Let $E = \{H, T\} =$ the event of occurrence of a head or a tail.

Then, E is a mixed event.

EQUALLY LIKELY EVENTS

A set of events is said to be equally likely if none of them is expected to occur in preference to the other.

For example, when a fair coin is tossed, then occurrence of head or tail are equally likely cases and there is no reason to expect a 'head' or a 'tail' in preference to the other.

EXHAUSTIVE EVENTS

A set of events is said to be exhaustive if the performance of the experiment always results in the occurrence of atleast one of them.

For example, when a die is thrown, then the events

$A_1 = \{1, 2\}$ and $A_2 = \{2, 3, 4\}$ are not exhaustive as we can get 5 as outcome of the experiment which is not the member of any of the events A_1 and A_2.

But, if we consider the events $E_1 = \{1, 2, 3\}$ and $E_2 = \{2, 4, 5, 6\}$ then the set of events E_1, E_2 is exhaustive.

MUTUALLY EXCLUSIVE EVENTS

A set of events is said to be mutually exclusive if occurrence of one of them precludes the occurrence of any of the remaining events.

Thus, $E_1, E_2, ..., E_n$ are mutually exclusive if and only if $E_i \cap E_j = \phi$ for $i \neq j$.

For example, when a coin is tossed, the event of occurrence of a head and the event of occurrence of a tail are mutually exclusive events.

INDEPENDENT EVENTS

Two events are said to be independent, if the occurrence of one does not depend on the occurrence of the other.

For example, when a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

COMPLEMENT OF AN EVENT

The complement of an event E, denoted by $\bar{E}$ or E' or E^c, is the set of all sample points of the space other than the sample points in E.

For example, when a die is thrown, sample space

$$S = \{1, 2, 3, 4, 5, 6\}.$$

If $E = \{1, 2, 3, 4\}$, then $\bar{E} = \{5, 6\}$.

Note that $E \cup \bar{E} = S$.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE EVENTS

A set of events $E_1, E_2, ..., E_n$ of a sample space S form a mutually exclusive and exhaustive system of events, if

(i) $E_i \cap E_j = \phi$ for $i \neq j$ and

(ii) $E_1 \cup E_2 \cup ... \cup E_n = S$.

For example, when a die is thrown, sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Let $E_1 = \{1, 3, 5\} =$ the event of occurrence of an odd number and

$E_2 = \{2, 4, 6\} =$ the event of occurrence of an even number.

Then, $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

PROBABILITY OF OCCURRENCE OF AN EVENT

Let S be a sample space, then the probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

$$= \frac{\text{number of cases favourable to event } E}{\text{total number of cases}}$$

Note :

(a) $0 \leq P(E) \leq 1$, i.e. the probability of occurrence of an event is a number lying between 0 and 1.

(b) $P(\phi) = 0$, i.e. probability of occurrence of an impossible event is 0.

(c) $P(S) = 1$, i.e. probability of occurrence of a sure event is 1.

ODDS IN FAVOUR OF AN EVENT AND ODDS AGAINST AN EVENT

If the number of ways in which an event can occur be m and the number of ways in which it does not occur be n, then

(i) odds in favour of the event $=\frac{m}{n}$ and

(ii) odds against the event $=\frac{n}{m}$.

If odds in favour of an event are $a:b$ then the probability of the occurrence of that event is $\frac{a}{a+b}$ and the probability of the non-occurrence of that event is $\frac{b}{a+b}$.

SOME IMPORTANT RESULTS ON PROBABILITY

1. $P(\overline{A})=1-P(A)$
2. If A and B are any two events, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.
3. If A and B are mutually exclusive events, then $A\cap B=\phi$ and hence $P(A\cap B)=0$.
   $\therefore\ P(A\cup B)=P(A)+P(B)$.
4. If A, B, C are any three events, then $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)$
   $-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$.
5. If A, B, C are mutually exclusive events, then $A\cap B=\phi,\ B\cap C=\phi,\ C\cap A=\phi,\ A\cap B\cap C=\phi$ and hence $P(A\cap B)=0,\ P(B\cap C)=0,\ P(C\cap A)=0,\ P(A\cap B\cap C)=0$.
   $\therefore\ P(A\cup B\cup C)=P(A)+P(B)+P(C)$.
6. $P(\overline{A}\cap\overline{B})=1-P(A\cup B)$.
7. $P(\overline{A}\cup\overline{B})=1-P(A\cap B)$.
8. $P(A)=P(A\cap B)+P(A\cap\overline{B})$
9. $P(B)=P(B\cap A)+P(B\cap\overline{A})$
10. If $A_1, A_2, ..., A_n$ are independent events, then $P(A_1\cap A_2\cap ...\cap A_n)=P(A_1)\cdot P(A_2),..., P(A_n)$.
11. If $A_1, A_2,..., A_n$ are mutually exclusive events, then
    $P(A_1\cup A_2\cup...\cup A_n)=P(A_1)+P(A_2)+...+P(A_n)$.
12. If $A_1, A_2,..., A_n$ are exhaustive events, then $P(A_1\cup A_2\cup...\cup A_n)=1$.
13. If $A_1, A_2,..., A_n$ are mutually exclusive and exhaustive events, then
    $P(A_1\cup A_2\cup...\cup A_n)=P(A_1)+P(A_2)+...+P(A_n)=1$.
14. If $A_1, A_2,..., A_n$ are n events, then
    (a) $P(A_1\cup A_2\cup...\cup A_n)\le P(A_1)+P(A_2)+...+P(A_n)$
    (b) $P(A_1\cap A_2\cap...\cap A_n)\ge 1-P(\overline{A}_1)-P(\overline{A}_2)...-P(\overline{A}_n)$

CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and $P(B)\neq 0$, is called the conditional probability and it is denoted by $P(A/B)$.

Thus $P\left(A/B\right)$ = Probability of occurrence of A, given that B has already happened.

$$= \frac{P(A\cap B)}{P(B)} = \frac{n(A\cap B)}{n(B)}$$

Similarly, $P\left(B/A\right)$ = Probability of occurrence of B, given that A has already happened.

$$= \frac{P(A\cap B)}{P(A)} = \frac{n(A\cap B)}{n(A)}.$$

Sometimes, $P\left(A/B\right)$ is also used to denote the probability of occurrence of A when B occurs. Similarly, $P\left(B/A\right)$ is used to denote the probability of occurrence of B when A occurs.

1. **Multiplication theorems on probability**

   (i) If A and B are two events associated with a random experiment, then $P(A\cap B) = P(A).P\left(B/A\right)$, If $P(A)\neq 0$ or $P(A\cap B) = P(B).P\left(A/B\right)$, if $P(B)\neq 0$

   (ii) **Extension of multiplication theorem :** If $A_1, A_2 ... A_n$ are n events related to a random experiment, then $P(A_1\cap A_2\cap A_3\cap ...\cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1\cap A_2)\ldots P(A_n/A_1\cap A_2\cap ..\cap A_{n-1})$. where $P(A_i/A_1\cap A_2\cap ..\cap A_{i-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2 ... A_{i-1}$ have already happened

   (iii) **Multiplication theorems for independent events:** If A and B are independent events associated with a random experiment, then $P(A\cap B) = P(A).P(B)$ *i.e* the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities. By multiplication theorem, we have $P(A\cap B) = P(A).P(B/A)$ .since A and B are independent events, therefore

   $P(B/A) = P(B)$. Hence , $P(A\cap B) = P(A).P(B)$.

   (iv) **Extension of multiplication theorem for independent events:** If $A_1, A_2 ... A_n$ are independent events associated with a random experiment, then $P(A_1\cap A_2\cap A_3\cap ...\cap A_n) = P(A_1)P(A_2)...P(A_n)$.
   By multiplication theorem, we have
   $$P(A_1\cap A_2\cap A_3\cap ...\cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1\cap A_2)...P(A_n/A_1\cap A_2\cap ...\cap A_{n-1}).$$
   Since $A_1, A_2 ... A_{n-1}, A_n$ are independent events, therefore
   $$P(A_2/A_1) = P(A_2), P(A_3/A_1\cap A_2) = P(A_3)...P(A_n/A_1\cap A_2\cap ...\cap A_{n\_1}) = P(A_n)$$
   Hence, $P(A_1\cap A_2\cap ...\cap A_n) = P(A_1)P(A_2)...P(A_n)$.

2. **Probability of at least one of the n independent events :**

   If $p_1, p_2, p_3 ... p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_3 ... A_n$ respectively, then

   (i) Probability of happening none of them = $P(\overline{A}_1\cap \overline{A}_2\cap \overline{A}_3 ...\cap \overline{A}_n) = P(\overline{A}_1).P(\overline{A}_2).P(\overline{A}_3)...P(\overline{A}_n)$

   $= (1-p_1)(1-p_2)(1-p_3)...(1-p_n)$

   (ii) Probability of happening at least one of them

   $= P(A_1\cup A_2\cup A_3 ...\cup A_n) = 1 - P(\overline{A_1})P(\overline{A_2})P(\overline{A_3})...P(\overline{A_n}). = 1-(1-p_1)(1-p_2)(1-p_3)...(1-p_n)$

   (iii) Probability of happening of first event and not happening of the remaining

   $= P(A_1)P(\overline{A_2})P(\overline{A_3})...P(\overline{A_n}) = p_1(1-p_2)(1-p_3)...(1-p_n)$

LAW OF TOTAL PROBABILITY

Let S be the sample space and let $E_1, E_2, ..., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or $...E_n$, then

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + ... + P(E_n) \cdot P\left(\frac{A}{E_n}\right)$$

BAYE'S RULE

Let S be a sample space and $E_1, E_2, ... E_n$ be n mutually exclusive events such that $\bigcup_{i=1}^{n} E_i = S$ and $P(E_i) > 0$ for $i = 1, 2, ..., n.$ We can think of E_i's as the causes that lead to the outcome of an experiment. The probabilities $P(E_i), i = 1, 2, ..., n$ are called prior probabilities. Suppose the experiment results in an outcome of event $A,$ where $P(A) > 0.$ We have to find the probability that the observed event A was due to cause $E_i,$ that is, we seek the conditional probability $P(E_i / A).$ These probabilities are called posterior probabilities, given by Baye's rule as $P(E_i / A) = \dfrac{P(E_i) \cdot P(A / E_i)}{\sum_{k=1}^{n} P(E_k) P(A / E_k)}.$

RANDOM VARIABLE

A random variable is a real valued function whose domain is the sample space of a random experiment.
A random variable is usually denoted by the capital letters $X, Y, Z, ...,$ etc.

DISCRETE RANDOM VARIABLE

A random variable which can take only finite or countably infinite number of values is called a discrete random variable.

CONTINUOUS RANDOM VARIABLE

A random variable which can take any value between two given limits is called a continuous random variable.

Geometrical method for probability: When the number of points in the sample space is infinite, it becomes difficult to apply classical definition of probability. For instance if we are interested to find the probabilities that a point selected at random from the interval [1,6] lies either in the interval [1,2] or [5,6], we cannot apply the classical definition of probability. In this case we define the probability as follows:

$$P\{x \in A\} = \frac{\text{Measure of region A}}{\text{Measure of the sample space S}}$$

where measure stands for length, area or volume depending upon whether S is a one – dimensional, two – dimensional or three dimensional region.

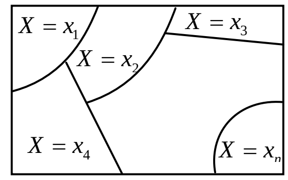
Probability distribution : Let S be a sample space. A random variable X is a function from the set S to R, the set of real numbers.

For example, the sample space for a throw of a pair of dice is

$$S = \begin{matrix} \{11, & 12, & \cdots, & 16 \\ 21, & 22, & \cdots, & 26 \\ \vdots & \vdots & \ddots & \vdots \\ 61, & 62, & \cdots, & 66\} \end{matrix}$$

Let X be the sum of numbers on the dice. Then $X(12) = 3, X(43) = 7,$ etc. Also, $\{X = 7\}$ is the event $\{61, 52, 43, 34, 25, 16\}.$ In general, if X is a random variable defined on the sample space S and r is a real number, then $\{X = r\}$ is an event.

If the random variable X takes n distinct values $x_1, x_2, ..., x_n$, then $\{X = x_1\}, \{X = x_2\}, ..., \{X = x_n\}$ are mutually exclusive and exhaustive events.



Now, since $(X = x_i)$ is an event, we can talk of $P(X = x_i)$.

If $P(X = x_i) = P_i \text{ (where } 1 \le i \le n)$, then the system of numbers.

$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ is said to be the probability distribution of the random variable X.

The expectation (mean) of the random variable X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$ and the variance of X is defined as $\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2$.

Bernoullian Trials, set of n trials is said to be Bernoullian if

(a) The value of n is finite i.e. number of trials are finite
(b) Each and every trial/experiment is independent
(c) Trial (Experiment) consist only two out comes namely success and failure.
(d) Probability of success & failure for each trial is fixed (same)

3. **Binomial probability distribution :** A random variable X which takes values $0, 1, 2, ..., n$ is said to follow binomial distribution if its probability distribution function is given by

$P(X = r) = {}^nC_r p^r q^{n-r}, \ r = 0, 1, 2, ..., n$

where $p, q > 0$ such that $p + q = 1$.

The notation $X \sim B(n, p)$ is generally used to denote that the random variable X follows binomial distribution with parameters n and p.

We have $P(X = 0) + P(X = 1) + ... + P(X = n)$.

$= {}^nC_0 p^0 q^{n-0} + {}^nC_1 p^1 q^{n-1} + ... + {}^nC_n p^n q^{n-n} = (q + p)^n = 1^n = 1$

Now probability of

(a) Occurrence of the event exactly r times $P(X = r) = {}^nC_r q^{n-r} p^r$.

(b) Occurrence of the event at least r times $P(X \ge r) = {}^nC_r q^{n-r} p^r + ... + p^n = \sum_{X=r}^{n} {}^nC_X p^X q^{n-X}$.

(c) Occurrence of the event at the most r times

$$P(0 \le X \le r) = q^n + {}^nC_1 q^{n-1} p + ... + {}^nC_r q^{n-r} p^r = \sum_{X=0}^{r} {}^nC_X \ p^X q^{n-X}.$$

Note :

1. If the probability of happening of an event in one trial be p, then the probability of successive happening of that event in r trials is p^r.
2. If n trials constitute an experiment and the experiment is repeated N times, then the frequencies of $0, 1, 2, ..., n$ successes are given by

$$N.P.(X = 0), \ N.P.(X = 1), \ N.P.(X = 2), \ ...., \ N.P.(X = n).$$

(i) **Mean and variance of the binomial distribution :** The binomial probability distribution is

X	0	1	2	....	n
$P(X)$	${}^nC_0 q^n p^0$	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$	....	${}^nC_n q^0 p^n$

The binomial probability distribution is $\sum_{i=1}^{n} X_i p_i = \sum_{X=1}^{n} X . {}^nC_X q^{n-X} p^X = np,$

The variance of the Binomial distribution is $\sigma^2 = npq$ and the standard deviation is $\sigma = \sqrt{(npq)}.$

(ii) **Use of multinomial expansion :** If a die has m faces marked with the numbers 1, 2, 3, ... m and if such n dice are thrown, then the probability that the sum of the number exhibited on the upper faces equal to p is given by the coefficient of x^p in the expansion of $\frac{\left(x + x^2 + x^3 + ... + x^m\right)^n}{m^n}.$

4. **Poission distribution :** It is the limiting case of B.D. under the following condition

(i) Number of trials are very-very large i.e. $n \to \infty$

(ii) $p \to 0$ (Here p is not exactly 0 but nearly approaches to zero)

(iii) $np = \lambda,$ a finite quantity (λ is called parameter)

$\to$ Probability of r success for poission distribution is given by

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!},\ r = 0, 1, 2, ...$$

$\to$ For poission distribution recurrence formula is given by

$$P(r+1) = \frac{\lambda}{r+1} P(r)$$

Note :

(a) **For Poission Distribution** mean $=$ variance $= \lambda = np.$

(b) If X and Y are independent poission variates with parameter λ_1 and λ_2 then $X + Y$ also has poission distribution with parameters $\lambda_1 + \lambda_2.$

5. **Normal Distribution**

For a normal distribution, number of trials are infinite.

The Normal probability function or distribution is given by

$$P(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ where } \frac{x-\mu}{\sigma} = z \text{ known as standard variate, } -\infty < x < \infty$$

Facts About Normal Distribution

(i) It is limiting case of B.D. i.e. $B(n, p)$ when $n \to \infty$

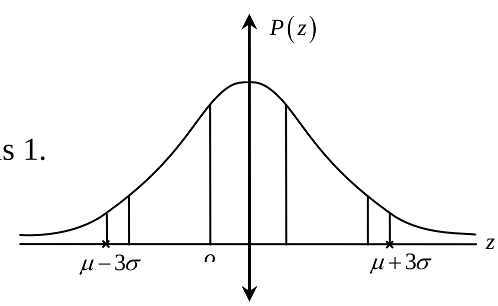
(ii) mean $=$ mode $=$ Median

(iii) Total area under a standard normal (curve) distribution is 1.

(iv) Normal curve is bell-shaped and uni-model

(v) $Q_3 -$ Median $=$ Median $- Q_1$

(vi) $\int_{-1}^{1} P(X = r)\, dx = 1$



μ = mean = mode = median

If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X - \mu}{\sigma}$ then $P(\mu - \sigma \le x \le \mu + \sigma) = P(-1 \le z \le 1) = .6827$

$P(\mu - 2\sigma \le x \le \mu + 2\sigma) = P(-2 \le z \le 2) = .9544$

$P(\mu - 3\sigma \le x \le \mu + 3\sigma) = P(-3 \le z \le 3) = .9973$

Note :

The probabilities $P(z_1 \le z \le z_2),\ P(z_1 < z < z_2)$
$P(z_1 < z \le z_2),\ P(z_1 < z < z_2)$ are all treated to be same.

Measures of Central Tendency and Dispersion

Chapter 34

INTRODUCTION

For given data, a single value of the variable representing the entire data is selected which described the characteristics of the data. Averages are, generally, the central part of the distribution and therefore, they are also called the **measures of central tendency**.

The following are the five measures of central tendency

(1) Arithmetic mean (2) Geometric mean (3) Harmonic mean

(4) Median (5) Mode

I. ARITHMETIC MEAN

Arithmetic mean is the most important among the mathematical mean.

According to **Horace Secrist**.

"*The arithmetic mean is the amount secured by dividing the sum of values of the items in a series by their number*."

1. **Simple arithmetic mean in individual series (Ungrouped data)**

(i) **Direct method**

If the series in this case be $x_1,\ x_2,\ x_3, ..., x_n$; then the arithmetic mean $\bar{x}$ is given by

$$\bar{x} = \frac{\text{Sum of the seires}}{\text{Number of terms}}$$

i.e., $$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} \quad = \frac{1}{n}\sum_{i=1}^{n} x_i$$

(ii) **Short cut method :** Arithmetic mean $(\bar{x}) = A + \frac{\sum d}{n}$,

where, $A =$ assumed mean,

$d =$ deviation from assumed mean $= x - A$, where x is the individual item,

$\sum d =$ sum of deviations and

$n =$ number of items.

2. **Simple arithmetic mean in continuous series (Grouped data)**

(i) **Direct method :** If the terms of the given series be $x_1,\ x_2,\ .....,x_n$ and the corresponding frequencies be $f_1,\ f_2,\ ....\ f_n$, then the arithmetic mean $\bar{x}$ is given by,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

(ii) **Short cut method :** Arithmetic mean $(\bar{x}) = A + \frac{\sum f(x - A)}{\sum f}$,

where $A =$ assumed mean,

$f =$ frequency and $x - A =$ deviation of each item from the assumed mean.

3. **Mean of the composite series :** If $\overline{x}_i, (i=1, 2, ..., k)$ are the means of k-component series of sizes $n_i, (i=1, 2, ..., k)$ respectively, then the mean $\overline{x}$ of the composite series obtained on combining the component series is given by the formula

$$\overline{x} = \frac{n_1\overline{x}_1 + n_2\overline{x}_2 + ... + n_k\overline{x}_k}{n_1 + n_2 + ... + n_k} = \frac{\sum_{i=1}^{n} n_i\overline{x}_i}{\sum_{i=1}^{n} n_i}.$$

4. **Properties of arithmetic mean**

(i) In a statistical data, the sum of the deviation of items from A.M. is always zero.

i.e., $\sum_{i=1}^{n} f_i\left(x_i - \overline{x}\right) = 0$, where f_i is the frequency of $x_i\,(1 \le i \le n)$.

(ii) In a statistical data, the sum of squares of the deviations of items A.M. is least

i.e., $\sum_{i=1}^{n} f_i\left(x_i - \overline{x}\right)^2$ is least.

(iii) If each of the n given observations be doubled, then their mean is doubled.

(iv) If $\overline{x}$ is the mean of $x_1, x_2, ..., x_n$ then the mean of $ax_1, ax_2, ..., ax_n$ where a is any number different from zero, is $a\,\overline{x}$.

II. GEOMETRIC MEAN

1. If $x_1, x_2, x_3, ..., x_n$ are n values of a variate x, none of them being zero, then geometric mean (G.M.) is given by

$$\text{G.M.} = \left(x_1 \cdot x_2 \cdot x_3 ..... x_n\right)^{1/n}$$

or $$\log\left(\text{G.M.}\right) = \frac{1}{n}\left(\log x_1 + \log x_2 + ... + \log x_n\right).$$

or $$\text{G.M.} = \text{antilog}\left(\frac{\log x_1 + \log x_2 + ... + \log x_n}{n}\right).$$

2. In case of frequency distribution, G.M. of n values $x_1, x_2, ... x_n$ of a variate x occurring with frequency $f_1, f_2, ..., f_n$ is given by

$$\text{G.M.} = \left(x_1^{f_1} \cdot x_2^{f_2} ... x_n^{f_n}\right)^{1/N}, \text{ where } N = f_1 + f_2 + ... + f_n.$$

or $$G.M. = \text{antilog}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{N}\right)$$

III. HARMONIC MEAN

1. The harmonic mean of n items $x_1, x_2, ..., x_n$ is defined as

$$\text{H.M.} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}}.$$

2. If the frequency distribution is $f_1, f_2, f_3, ..., f_n$ respectively,

then H.M. $=\dfrac{f_1+f_2+f_3+...+f_n}{\dfrac{f_1}{x_1}+\dfrac{f_2}{x_2}+...+\dfrac{f_n}{x_n}}$.

RELATION BETWEEN A.M. AND H.M.

The arithmetic mean (A.M.), geometric mean (G.M.) and harmonic mean (H.M.) for a given set of observation of a series are related as under :

$$\text{A.M.} \ge \text{G.M.} \ge \text{H.M.}$$

Equality sign holds only when all the observation in the series are same.

IV. MEDIAN

Median is the middle most or the central value of the variate in a set of observations, when the observations are arranged either in ascending or in descending order of their magnitudes. It divides the arranged series in two equal parts.

1. **CALCULATION OF MEDIAN**

(i) **Individual series :** If the data is raw, arrange in ascending or descending order. Let n be the number of observations.

If n is odd, Median $=$ value of $\left(\dfrac{n+1}{2}\right)th$ item.

If n is even, Median $=\dfrac{1}{2}\left[\text{value of }\left(\dfrac{n}{2}\right)th\text{ item } + \text{ value of }\left(\dfrac{n}{2}+1\right)th\text{ item}\right]$

(ii) **Discrete series :** In this case, we first find the cumulative frequencies of the variables arranged in ascending or descending order and the median is given by

Median $=\left(\dfrac{n+1}{2}\right)th$ observation, where n is the cumulative frequency.

(iii) **For grouped or continuous series :** In this case, following formula can be used.

(a) For series in ascending order, Median $=l+\dfrac{\left(\dfrac{N}{2}-C\right)}{f}\times i$

Where l = Lower limit of the median class

f = Frequency of the median class

N = The sum of all frequencies

i = The width of the median class

C = The cumulative frequency of the class preceding to median class.

(b) For series in descending order

Median $=u-\left(\dfrac{\dfrac{N}{2}-C}{f}\right)\times i,$ where $u=$ upper limit of the median class, $N=\sum_{i=1}^{n} f_i$.

As median divides a distribution into two equal parts, similarly the quartiles, quantiles, deciles and percentiles divide the distribution respectively into $4, 5, 10$ and 100 equal parts.

The j^{th} quartile is given by

$$Q_j = l + \left(\frac{j\frac{N}{4} - C}{f} \right) i \; ; \qquad j = 1,\ 2,\ 3.$$

Q_1 is the lower quartile,

Q_2 is the median and

Q_3 is called the upper quartile.

2. **LOWER QUARTILE**

(i) **Discrete series**

$$Q_1 = \text{ size of } \left(\frac{n+1}{4} \right) th \text{ item}$$

(ii) **Continuous series**

$$Q_1 = l + \frac{\left(\frac{N}{4} - C \right)}{f} \times i$$

3. **UPPER QUARTILE**

(i) **Discrete series**

$$Q_3 = \text{ size of } \left[\frac{3(n+1)}{4} \right] th \text{ item}$$

(ii) **Continuous series**

$$Q_3 = l + \frac{\left(\frac{3N}{4} - C \right)}{f} \times i$$

4. **DECILE**

Decile divide total frequencies N into ten equal parts.

$$D_j = l + \frac{\frac{N \times j}{10} - C}{f} \times i, \qquad [\text{where } j = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\ ]$$

5. **PERCENTILE**

Percentile divide total frequencies N into hundred equal parts and

$$P_k = l + \frac{\frac{N \times k}{100} - C}{f} \times i, \qquad \text{where } k = 1,\ 2,\ 3,\ 4,\ 5,\ ......,\ 99.$$

V. MODE

Mode is that value in a series which occurs most frequently. In a frequency distribution, mode is that variate which has the maximum frequency .

For continuous series, mode is calculated as,

$$\text{Mode } = l_1 + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right] \times i$$

Where, l_1 = The lower limit of the model class

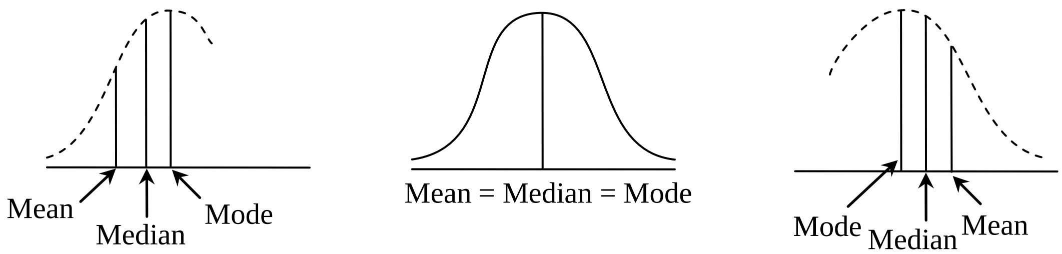
f_1 = The frequency of the model class

f_0 = The frequency of the class preceding the model class

f_2 = The frequency of the class succeeding the model class
i = The size of the model class.

SYMMETRIC DISTRIBUTION

A distribution is a symmetric distribution if the values of mean, mode and median coincide. In a symmetric distribution frequencies are symmetrically distributed on both sides of the centre point of the frequency curve.



A distribution which is not symmetric is called a skewed-distribution. In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., we have the following empirical relation between them,

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median}) \;\Rightarrow\; \text{Mode} = 3\,\text{Median} - 2\,\text{Mean}.$$

It is known as **Empirical relation**.

PIE CHART (PIE DIAGRAM)

In this diagram each item has a sector whose area has the same percentage of the total area of the circle as this item has of the total of such items. For example, if N be the total and n_1 is one of the components of the figure corresponding to a particular item, then the angle of the sector for this item $= \left(\frac{n_1}{N}\right) \times 360°$, as the total number of degree in the angle subtended by the whole circular arc at its centre is $360°$.

MEASURE OF DISPERSION

The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four measures of dispersion are

1. Range 2. Mean deviation 3. Standard deviation 4. Square deviation

1. **Range**

It is the difference between the values of extreme items in a series. Range $= X_{\max} - X_{\min}$

The coefficient of range (scatter) $= \dfrac{X_{\max} - X_{\min}}{X_{\max} + X_{\min}}$.

Range is not the measure of central tendency.Range is commonly used measures of dispersion in case of changes in interest rates, exchange rate, share prices and like statistical information.

(i) **Inter-quartile range :** We know that quartiles are the magnitudes of the items which divide the distribution into four equal parts. The inter-quartile range is found by taking the difference between third and first quartiles and is given by the following formula,

Inter-quartile range $= Q_3 - Q_1$,

where Q_1 = First quartile or lower quartile and Q_3 = Third quartile or upper quartile.

(ii) **Percentile range :** This is measured by the following formula,

Percentile range $= P_{90} - P_{10}$,

where $P_{90} = 90$ th percentile and $P_{10} = 10$ th percentile.

Percentile range is considered better than range as well as inter-quartile range.

(iii) **Quartile deviation or semi inter-quartile range :** It is one-half of the difference between the third quartile and first quartile i.e. $Q.D. = \dfrac{Q_3 - Q_1}{2}$

and coefficient of quartile deviation $= \dfrac{Q_3 - Q_1}{Q_3 + Q_1}$,

where Q_3 is the third or upper quartile and Q_1 is the lowest or first quartile.

2. **Mean deviation**

The arithmetic average of the deviations (all taking positive) from the mean, median or mode is known as mean deviation.

(i) **Mean deviation from ungrouped data (or individual series)**

Mean deviation $= \dfrac{\sum |x - M|}{n}$,

where $|x - M|$ means the modulus of the deviation of the variate from the mean (mean, median or mode) and n is the number of terms.

(ii) **Mean deviation from continuous series :**

Mean deviation $= \dfrac{\sum f |x - M|}{n} = \dfrac{\sum f\, dM}{n}$,

where $n = \sum f$. and $dM = |x - M|$ the deviation of each variate from the mean M .

3. **Standard deviation**

Standard deviation (or S.D.) is the square root of the arithmetic mean of the square of deviations of various values from their arithmetic mean and is generally denoted by σ read as sigma.

(i) **Coefficient of standard deviation :** To compare the dispersion of two frequency distributions the relative measure of standard deviation is computed which is known as coefficient of standard deviation and is given by

Coefficient of S.D. $= \dfrac{\sigma}{\overline{x}}$, where $\overline{x}$ is the A.M.

(ii) **Standard deviation from individual series**

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{N}}$$

where, $\overline{x}$ = The arithmetic mean of series

N = The total frequency.

(iii) **Standard deviations from continuous series**

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \overline{x})^2}{N}}$$

where, $\overline{x}$ = Arithmetic mean of series

x_i = Mid value of the class

f_i = Frequency of the corresponding x_i

N = $\sum f$ = The total frequency

Short cut method :

(i) $\sigma = \sqrt{\dfrac{\sum fd^2}{N} - \left(\dfrac{\sum fd}{N}\right)^2}$ (ii) $\sigma = \sqrt{\dfrac{\sum d^2}{N} - \left(\dfrac{\sum d}{N}\right)^2}$

where, $d = x - A =$ Deviation from the assumed mean A

$f =$ Frequency of the item

$N = \sum f =$ Sum of frequencies

4. **Square deviation**

(i) **Root mean square deviation**

$$S = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i\left(x_i - A\right)^2}.$$

where A is any arbitrary number and S is called root mean square deviation.

(ii) **Relation between S.D. and root mean square deviation :** If σ be the standard deviation and S be the root mean square deviation.

Then, $S^2 = \sigma^2 + d^2$.

Obviously, S^2 will be least when $d = 0$ i.e., $\bar{x} = A$

Hence, mean square deviation and consequently root mean square deviation is least, if the deviations are taken from the mean.

VARIANCE

The square of standard deviation is called the variance.

Coefficient of standard deviation and variance : The coefficient of standard deviation is the ratio of the S.D. to A.M. i.e., $\frac{\sigma}{x}$.

Coefficient of variance $=$ coefficient of S.D. $\times 100 = \frac{\sigma}{\bar{x}} \times 100$.

Variance of the combined series : If n_1, n_2 are the sizes, $\bar{x}_1, \bar{x}_2$ the means and σ_1, σ_2 the standard deviation of two series, then $\sigma^2 = \frac{1}{n_1 + n_2}\left[n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)\right]$,

where $d_1 = \bar{x}_1 - \bar{x}$, $d_2 = \bar{x}_2 - \bar{x}$ and $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

SKEWNESS

"Skewness" measures the lack of symmetry. It is measured by $\gamma_1 = \frac{\sum\left(x_i - \mu\right)^3}{\left\{\sum\left(x_i - \mu^2\right)\right\}^{3/2}}$ and is denoted by γ_1.

The distribution is skewed if,

(i) Mean $\neq$ Median $\neq$ Mode

(ii) Quartiles are not equidistant from the median

(iii) The frequency curve is stretched more to one side than to the other.

1. **Distribution :** There are three types of distributions.

(i) **Normal distribution :** When $\gamma_1 = 0$, the distribution is said to be normal.

In this case, Mean $=$ Median $=$ Mode

(ii) **Positively skewed distribution :** When $\gamma_1 > 0$, the distribution is said to be positively skewed.

In this case, Mean $>$ Median $>$ Mode

(iii) **Negative skewed distribution :** When $\gamma_1 < 0$, the distribution is said to be negatively skewed.

In this case, Mean $<$ Median $<$ Mode

2. **Measures of skewness**

(i) **Absolute measures of skewness :** Various measures of skewness are

(a) $S_K = M - M_d$ (b) $S_K = M - M_o$ (c) $S_K = Q_3 + Q_1 - 2M_d$

where, M_d = median, M_o = mode, M = mean.

Absolute measures of skewness are not useful to compare two series, therefore relative measure of dispersion are used, as they are pure numbers.

3. **Relative measures of skewness**

(i) **Karl Pearson's coefficient of skewness**

$$S_k = \frac{M - M_o}{\sigma} = 3\frac{(M - M_d)}{\sigma}, \ -3 \le S_k \le 3$$ where σ is standard deviation.

(ii) **Bowley's coefficient of skewness**

$$S_k = \frac{Q_3 + Q_1 - 2M_d}{Q_3 - Q_1}$$

Bowley's coefficient of skewness lies between -1 and 1.

(iii) **Kelly's coefficient of skewness**

$$S_K = \frac{P_{10} + P_{90} - 2M_d}{P_{90} - P_{10}} = \frac{D_1 + D_9 - 2M_d}{D_9 - D_1}.$$

Correlation and Regression **Chapter 35**

CORRELATED VARIABLES

Two variables are said to be correlated if the change in one quantity followed by the change in the other quantity.

TYPE OF CORRELATION

(i) **Positive Correlation (Direct Correlation) :** Correlation is said to be positive when the increase in the value of one variable is accompanied by an increase in the value of other variable and vice-versa.

(ii) **Negative or Inverse Correlation :** Correlation is said to negative if increase in the value of one variable is accompanied by an decrease in the value of the other variable and vice-versa.

(iii) **Linear Correlation :** When one variable moves with the other variable in some fixed quantity or fixed properties, such relation are shown by a straight line in the graph.

(iv) **Non-linear Correlation :** When the change in one variable does not bear a constant ratio with the change of other variable then it is called noon-linear correlation.

CO-VARIANCE

It denote the degree of inter-dependence between the variable quantities, Co-variance between the two quantities is denoted by $Cov\left(x,\ y\right)$ and defined by (if data's are individual).

$$Cov\left(x,\ y\right)=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i y_i-\bar{x}\,\bar{y} \qquad =\frac{1}{n}\sum_{i=1}^{n}x_i y_i-\left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right).$$

METHOD OF FINDING CORRELATION

(i) **Scatter diagram**

With the help of scatter diagram we can develop the relationship .i.e., correlation between the variable quantities by taking one of them on $x-axis$ and other on $y-axis$, using all distinct order pair of these two variable quantities we get a picture in the form of dots, due to this reason. We called it dot diagram also. If these dots lies in the same straight line then with the help of the direction it is said to $+ve$ correlation and for obtuse angle it is said to be negative correlation.

(ii) **Karl Pearson's co-efficient of correlation**

We denote coefficient of correlation by r_{xy}.

$$\therefore \quad r_{xy}=\frac{\text{Cov}\left(\text{x,y}\right)}{\sigma_x\sigma_y} \quad \text{or} \quad =\frac{\text{Cov}\left(x,y\right)}{\sqrt{\text{Var }x}\sqrt{\text{Var }y}} \quad \text{or} \quad =\frac{\text{Cov}\left(x,y\right)}{\text{SD}_\text{x}.\text{SD}_\text{y}}$$

where σ_x,σ_y are standard deviation (SD) of variables x and y respectively.

$$\sigma_x{}^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2 \qquad \ldots\text{(i)}$$

$$=\frac{1}{n}\sum^{n}x_i{}^2-\left(\frac{\sum x_i}{n}\right)^2 \qquad \ldots\text{(ii)}$$

$$=\frac{1}{n}\sum d^2{}_x-\left(\frac{\sum d_x}{n}\right)^2 \qquad \ldots\text{(iii)}$$

Now $r_{xy} = \dfrac{\text{Cov}(x,y)}{\sigma_x \sigma_y}$ …(i)

$$= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} \quad \text{…(ii)}$$

$$= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}} = \frac{n\sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{\sqrt{\left[n\sum x_i^2 - \left(\sum x_i\right)^2\right] \times \left[n\sum y_i^2 - \left(\sum y_i\right)^2\right]}}$$

Shortcut method

$$r_{xy} = \frac{n\sum (d_x d_y) - \left(\sum d_x\right)\left(\sum d_y\right)}{\sqrt{n\sum d_x^2 - \left(\sum d_x\right)^2} \times \sqrt{n\sum d_y^2 - \left(\sum d_y\right)^2}} \quad \text{…(iv)}$$

where $n =$ the number of paired observation. $\begin{matrix} dx = x - A \\ dy = x - B \end{matrix} \left(\begin{matrix} A, B \text{ are assumed mean} \\ \text{for series } x \text{ and } y, \text{respectively.} \end{matrix} \right)$

(iii) **Rank of correlation (Spearman's coefficient of correlation)**

Let (x_i, y_i) be set of n ordered pairs of observation. Let d be the different between paired ranks. We denote the rank correlation by ρ and defined by as

$$\rho = 1 - 6\frac{\sum d^2}{n(n^2 - 1)} \text{ where } d = R_1 - R_2$$

PROPERTIES OF CORRELATION

(i) The values of r always lies between -1 and $1,$ including itself the numbers $-1,$ and $1.$
i.e., $-1 \le r_{xy} \le 1$ or $-1 \le r \le 1.$

(ii) If $0 < r < 1$ known as positive correlation.

(iii) If $0 < r < .5,$ known as low positive correlation.

(iv) If $0.5 < r < 1,$ known as high positive correlation.

(v) If $r = +1,$ known as perfect positive correlation.

(vi) If $r = -1,$ known as perfect negative correlation.

(vii) If $-0.5 < r < 0,$ known as low negative correlation.

(viii) If $-1 < r < -0.5,$ known as high negative correlation.

(ix) If $r = 0$ then x, y are said to uncorrelated but it does not indicate that x, y are independent.

(x) If x, y are two independent variable then $r(x, y) = 0.$

(xi) The value of r_{xy} .i.e., coefficient to correlation is independent of the change in origin and scale.

(xii) Coefficient of correlation is purely a number which has no unit of measurement.

(xiii) If the variables are connected by the equation $Ax + By + k = 0$ then,

$$r_{xy} = -1 \quad \text{if} \quad AB > 0 \quad \text{and}$$

$$\gamma_{xy} = 1 \quad \text{if} \quad AB < 0$$

(xiv) For three variables x, y, z if $z = Ax + By$ then

$\sigma_z^2 = A^2\sigma_x^2 + B^2\sigma_y^2 + (2AB\sigma_x\sigma_y).r,$ where $r =$ coefficient of correlation between x and y.

(xv) For variable x and y

$$r\left(Ax+k_1, By+k_2\right)=\frac{|AB|}{AB}r\left(x,y\right) \text{ such that } A,B\neq 0$$

DEDUCTIONS

(i) Standard error

$$\left(SE\left(r\right)\right)=\frac{1-r^2}{\sqrt{n}} \qquad r=r_{xy},$$

$n=$ Number of observation.

(ii) $\dfrac{PE\left(r\right)}{SE\left(r\right)}=\dfrac{\text{Probable error}}{\text{Standerd error}}=.6745$

(iii) $r<PE\left(r\right)$, No evidence of r.

(iv) $r>6\,PE\left(r\right)$, existence of correlation is definite

(v) r^2 is known as coefficient of determination.

REGRESSION ANALYSIS

According to British scientist "*Sir Francis Galton*" Regression means stepping back towards the average. Regression may be linear or curvilinear.

Equation of Regression lines

Line of regression y on x

$$y-\overline{y}=r\frac{\sigma_y}{\sigma_x}\left(x-\overline{x}\right) \quad =b_{yx}\left(x-\overline{x}\right)$$

Line of regression x on y

$$x-\overline{x}=r\frac{\sigma_x}{\sigma_y}\left(y-\overline{y}\right) \quad =b_{xy}\left(y-\overline{y}\right)$$

Where b_{yx} and b_{xy} are regression coefficient y on x and y respectively and defined as

$$b_{yx}=r\frac{\sigma_y}{\sigma_x} \quad \text{and} \quad b_{xy}=r\frac{\sigma_x}{\sigma_y}$$

Facts of regression lines:

(i) Two regression equations intersects at mean values of the data's $\left(x_i,y_i\right)$ *.i.e.,* regression lines intersect at $\overline{x},\overline{y}$.

(ii) To determine $\overline{x},\overline{y},$ solve the two regression lines.

(iii) The sign. of b_{yx},b_{xy} and r are always same.

(iv) The regression line y on x is used to estimate the value of y for the given value of the variables x Here y is dependent and x is independent variable.

(v) The regression line x on y is used to determined the value of x for the given value of y. In this case x is dependent on y.

Facts of regression coefficients

(i) The sign of b_{yx},b_{yx} and coefficient of correlation are either both $+ve$ or both negative.

(ii) $r_{xy}=\sqrt{b_{yx}.b_{xy}}$ i.e., r is $G.M.$ between regression coefficient.

$\therefore\ r^2=b_{yx}b_{xy}$

(iii) $0<\left|\ b_{xy}b_{yx}\ \right|\leq 1$ if $r\neq 0$, if $\left|\ b_{xy}\right|>1$ then $\left|\ b_{yx}\right|<1$.

(iv) If $b_{yx}, b_{xy}, r > 0$ then $\frac{1}{2}\left(b_{xy} + b_{yx}\right) \geq r_{xy}$.

(v) If $b_{yx}, b_{xy}, r_{xy} < 0$ then $\frac{1}{2}\left(b_{xy} + b_{yx}\right) \leq r_{xy}$.

(vi) Regression coefficient are independent of the change of origin but not of scale.

Angle between the regression lines

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \quad \ldots(i)$$

$$= \frac{\left(1 - r^2\right)}{|r|} \frac{\sigma_x . \sigma_y}{\sigma^2{}_x + \sigma^2{}_y} \quad \ldots(ii)$$

$$= \frac{b_{yx} . b_{xy} - 1}{b_{yx} + b_{xy}} \quad \ldots(iii)$$

where θ be the angle between the lines of regression.

$b_{yx} = \frac{r\sigma_y}{\sigma_x} = m_1 =$ slope of regression line y on x

$b_{xy} = r\frac{\sigma_x}{\sigma_y} = m_2 =$ slope of regression line x on y .

(i) $$\sin^2\theta = \frac{\left(1 - r^2\right)^2}{\left(1 - r^2\right)^2 + r^2\left(\frac{\sigma^2{}_x + \sigma^2{}_y}{\sigma_x . \sigma_y}\right)^2} \qquad \therefore \qquad \sin^2\theta = \left(1 - r^2\right)^2$$

$\therefore \ \sin\theta \leq 1 - r^2$

(ii) If $m_1 m_2 = -1$ .i.e., regression lines are $\perp$ to each other.

Then $r = 0, \theta = \frac{\pi}{2}$ In this line y on x parallel to $x - axis$ and line x on y parallel to $y - axis$

(iii) If $\theta = 0, \pi$ The lines of regression are coincident, overlapping symmetrical or identical

$\therefore \qquad b_{yx} b_{xy} = 1, \ \therefore \ \ b_{yx} = \frac{1}{b_{xy}}$

(iv) If $a, b, c, d, \in R$ $\left(a, c \text{ having opposite sign}\right), r$ is coefficient of correlation between x and y then coefficient of correlation between $ax + b$ and $cx + d$ is given by. $r\left(ax + b, cx + d\right) = -r_{xy}$

(i) If r, b_{yx}, b_{xy} are positive then $\frac{1}{b_{yx}} + \frac{1}{b_{xy}} \geq \frac{2}{r}$

(ii) If the two variates x and y are having perfect correlation between them, then they will be connected by $\frac{x}{a} + \frac{y}{b} = 1$ where $a = \frac{\sigma_x \overline{y} - \sigma_y \overline{x}}{\sigma_y}$

$b = \frac{\sigma_x \overline{y} - \sigma_y \overline{x}}{\sigma_x}$, Standard error estimate of x is given by $S_n = \sigma_x \sqrt{1 - r^2}$ and Standard error of estimate of y is given by $S_y = \sigma_y \sqrt{1 - r^2}$

Statics **Chapter 36**

DEFINITIONS

Matter

Anything that occupies space and is perceived by our senses is matter. Table, cup, water, etc. are example of matter.

Body

A body is a portion of matter occupying finite space. It has, therefore, a definite volume and a definite mass.

Particle

A particle is a body indefinitely small in size, so that the distance between its different parts are negligible. It may be regarded as a mathematical point associated with mass

Rigid Body

A body is said to be rigid when it does not change its shape and size when subjected to external forces *i.e.* a rigid body is a body the distance between any two points of which always remains the same .

Force

Force is an agent which changes or tends to change the state of rest or uniform motion of a body.

Note :

Force is a vector quantity

REPRESENTATION OF A FORCE

A force is completely known if we know the following data about it :

(i) its magnitude

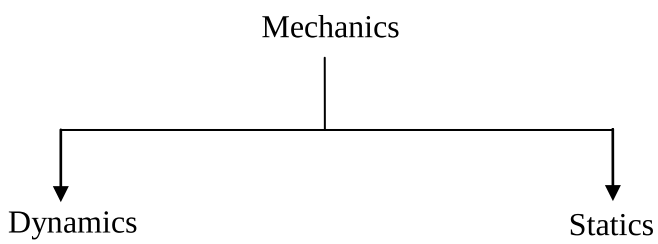
(ii) its direction

(iii) its point of application.

Thus, we can completely represent a force by a straight line *AB* drawn through the point of application along the line of action of the force, the length of the line *AB* representing the magnitude of the force and the order of the letters *A*, *B* specifying the direction.

MECHANICS

It is the science which deals with moving bodies or bodies at rest under the action of some forces.



DYNAMICS

It is that branch of mechanics which deals with the action of forces on bodies in motion.

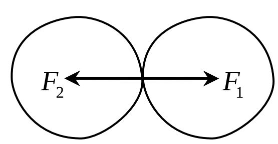
STATICS

It is that branch of mechanics which deals with the action of forces on bodies, the forces being so arranged that the bodies are at rest.

FORCES IN STATICS

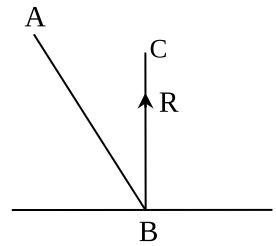
1. **Action and Reaction**

Whenever one body is in contact with another body, they apply equal and opposite forces at the point of contact. Such pair forces are called action and reaction pairs $\vec{F}_2 = -\vec{F}_1$

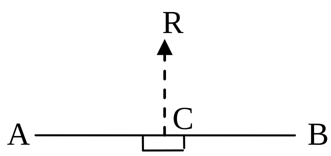


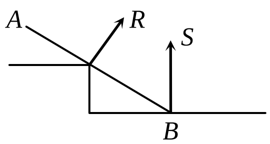
Some Important Cases

(i) When a rod AB rests with one end B upon a smooth plane, the reaction is along the normal to the plane at the point of contact.

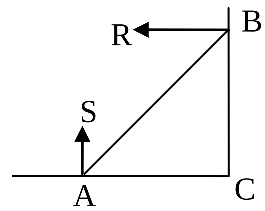


(ii) When a rod rests over a smooth peg, the reaction at the point of contact is $\perp$ to the rod.

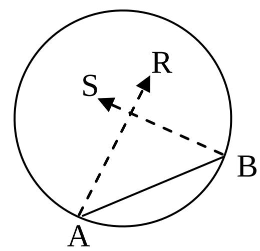




(iii) If one point of a body is in contact with the surface of another body, the reaction at the point of contact is $\perp$ to the surface, *e.g.* the equilibrium of a ladder in contact with the ground and a wall [both being smooth].



(iv) when a rod rests completely within a hollow sphere, the reactions at the extremities of the rod are along the normals at those points and will pass through centre of the hollow sphere .



2. **Weight**

Everybody is attracted towards the centre of the earth with a force proportional to its mass (the quantity of matter in the body).This force is called the **weight** of the body. If m is the mass of the body and g, is the acceleration due to gravity, then its weight $W = mg$.

3. **Tension or Thrust**

Whenever a string is used to support a weight or drag a body, there is a force of pull along the string. This force is called tension. Similarly, if some rod be compressed, a force will be exerted. This type of force is called thrust.

Note :

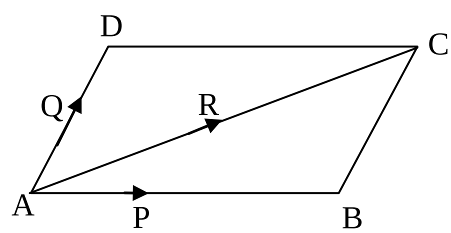
(i) The tension in a string is the same throughout. When two string are knotted together, the tensions in the two portions are different.

(ii) When a weight W hangs by a string, the tension in the string must be equal to the weight suspended. *i.e.* $T = W$.

(iii) The tension of a string always acts in a direction diverging away from the body under consideration and acts along the string.

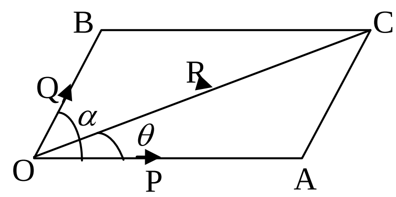
PARALLELOGRAM LAW OF FORCES

If two forces acting at a point , be represented in magnitude and direction by the sides of a parallelogram drawn from the point, their resultant is represented both in magnitude and direction, by the diagonal of the parallelogram drawn through that point. Let P and Q be the forces represented in magnitude and direction by the sides



AB and AD of a parallelogram ABCD, then their resultant R is represented in magnitude and direction by the diagonal AC.

RESULTANT OF TWO FORCES



If two concurrent forces P and Q are inclined at an angle α to each other, then the magnitude R of their resultant is given by $R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$. If R makes an angle θ with the direction of P, then $\tan\theta=\dfrac{Q\sin\alpha}{P+Q\cos\alpha}$

PARTICULAR CASES

1. When P and Q are at right angles to each other *i.e.* $\alpha=90°$

   In this case $R=\sqrt{P^2+Q^2}$ and $\tan\theta=\dfrac{Q}{P}$.

2. When $P=Q$. In this case, $R=2P\cos\dfrac{\alpha}{2}$ *and* $\theta=\dfrac{\alpha}{2}$.

3. When P and Q are in the same direction $i.e. \alpha=0$

   In this case, R is in the same direction as P and Q and $R=P+Q$.

   This is called the greatest resultant of the two forces.

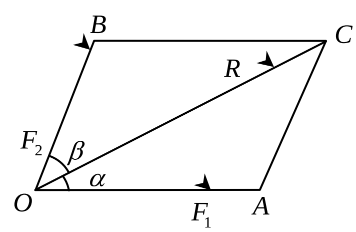
4. When P and Q are in the opposite direction $(i.e. \alpha=180°)$ and $P>Q$.

   In this case, R is in the direction of P and $R=P-Q$.

   This is called the least resultant of the two forces.

COMPONENT OF A FORCE IN TWO DIRECTIONS :

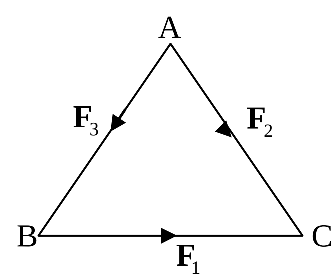
The component of a force R in two directions making angles α and β with the line of action of R on and opposite sides of it are



$$F_1=\frac{OC.\sin\beta}{\sin(\alpha+\beta)}=\frac{R\sin\beta}{\sin(\alpha+\beta)}$$

and $$F_2=\frac{OC.\sin\alpha}{\sin(\alpha+\beta)}=\frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

TRIANGLE LAW OF FORCES

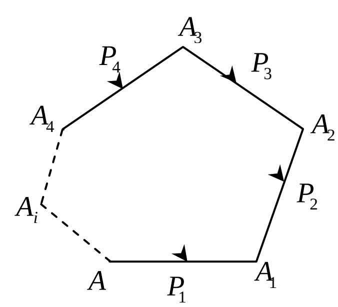


If three forces, acting at a point, be represented in magnitude and direction by the three sides of triangle, taken in order, they are in equilibrium.

$$\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3=0$$

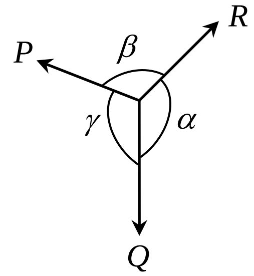
POLYGON LAW OF FORCES

If any number of forces acting on a particle be represented in magnitude and direction by the sides of a polygon taken in order, the forces shall be in equilibrium.



$$\mathbf{P}_1+\mathbf{P}_2+\ldots\ldots+\mathbf{P}_n=0$$

LAMI'S THEOREM

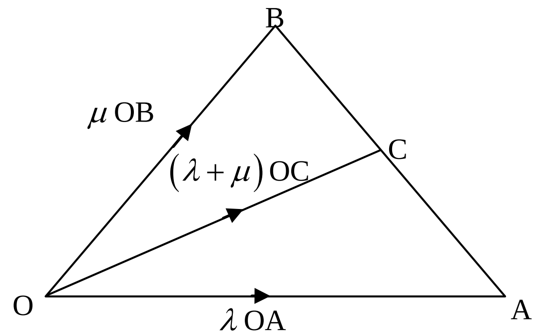


If three forces acting at a point be in equilibrium, each force is proportional to the sine of the angle between the other two. Thus if the forces are P, Q and R; α, β, γ be the angle between Q and R, R and P, P and Q respectively, also the forces are in equilibrium, we have,

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}.$$

The converse of this theorem is also true.

$\lambda - \mu$ THEOREM



The resultant of two forces, acting at a point *O* along *OA* and *OB* and represented in magnitude λ . *OA* and μ . *OB*, is represented by a force $(\lambda+\mu).OC$, where C is a point on AB such that $\lambda CA = \mu CB$ *i.e.C* divides AB in the ratio $\mu : \lambda$. In vector notation the above statement can be written as : $\lambda.\overrightarrow{OA} + \mu.\overrightarrow{OB} = (\lambda+\mu).\overrightarrow{OC}$, where C is a point on AB dividing it in the ratio $\mu : \lambda$.

Note :

In the above theorem, if $\lambda = \mu = 1$, then $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{2OC}$, where C is the mid point of AB ,*i.e.* the resultant of two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is $2\overrightarrow{OC}$, where C is the mid point of AB.

EQUILIBRIUM OF FORCES

A system of forces acting on a body is said to be in equilibrium if it produces no change in the motion of the body i.e.

(i) Vector sum of all forces is equal to zero and

(ii) Vector sum of all the moments of these forces about any point is zero.

EQUILIBRIUM OF TWO FORCES

Two forces acting at a point are in equilibrium if and only if they,

(i) are equal in magnitude

(ii) act along the same line

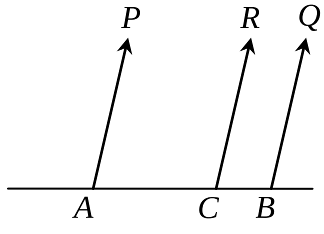
(iii) have opposite directions.

CONDITION OF EQUILIBRIUM OF A NUMBER OF COPLANAR CONCURRENT FORCES

A given number of forces acting at a point are in equilibrium if and only if the algebraic sum of their resolved parts in each of the two perpendicular directions OX and OY vanish separately.

PARALLEL FORCES

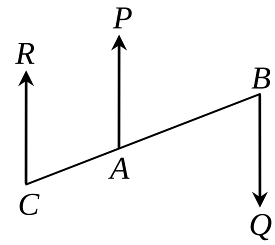
1. **Like parallel forces :** Two parallel forces are said to be like parallel forces when they act in the same direction.



The resultant R of two like parallel forces P and Q is equal in magnitude of the sum of the magnitudes of forces and R acts in the same direction as the forces P and Q and at the point on the line segment joining the point of action P and Q, which divides it in the ratio $Q : P$ internally.

2. **Two unlike parallel forces**

Two parallel forces are said to be unlike if they act in opposite directions.

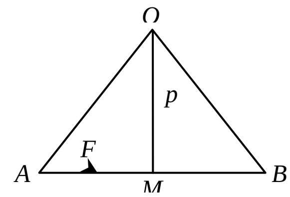


If P and Q be two unlike parallel forces acting at A and B and P is greater in magnitude than Q. Then their resultant R acts in the same direction as P and acts at a point C on BA produced. Such that $R = P - Q$ and $P.CA = Q.CB$

Then in this case C divides BA externally in the inverse ratio of the forces,

$$\frac{P}{CB} = \frac{Q}{CA} = \frac{P-Q}{CB-CA} = \frac{R}{AB}.$$

MOMENT



The moment of a force about a point O is given in magnitude by the product of the forces and the perpendicular distance of O from the line of action of the force. If F be a force acting at point A of a rigid body along the line AB and $OM\,(=P)$ be the perpendicular distance of the fixed point O from AB, then the moment of force about O.

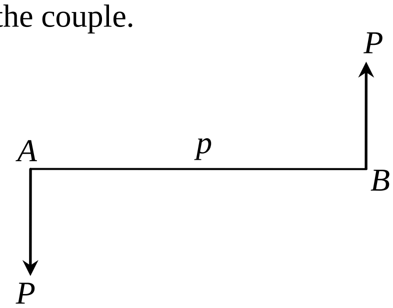
$$= F.p = AB \times OM = 2\left[\frac{1}{2}(AB \times OM)\right] = 2(\text{Area of } \Delta\, AOB)$$

The S.I. unit of moment is Newton-meter *(N-m).*

COUPLES

Two equal unlike parallel forces which do not have the same line of action, are said to form a couple.

1. **Arm of the couple :** The perpendicular distance between the lines of action of the forces forming the couple is known as the arm of the couple.

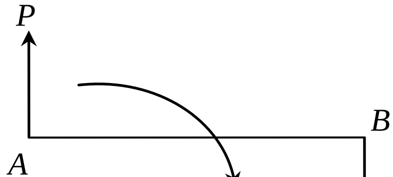


2. **Moment of couple**

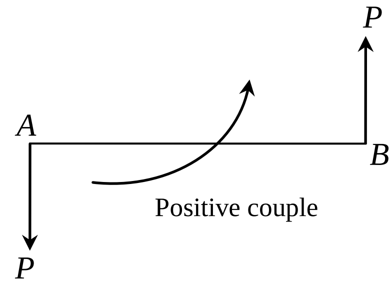
The moment of a couple is obtained in magnitude by multiplying the magnitude of one of the forces forming the couple and perpendicular distance between the lines of action of the force. The perpendicular distance between the forces is called the arm of the couple. The moment of the couple is regarded as positive or negative according as it has a tendency to turn the body in the anticlockwise or clockwise direction.

Moment of a couple $=$ Force $\times$ Arm of the couple $= P.p$

3. **Sign of the moment of a couple :** The moment of a couple is taken with positive or negative sign according as it has a tendency to turn the body in the anticlockwise or clockwise direction.



Negative couple

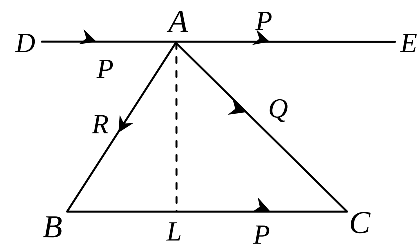


Positive couple

Note : A couple can not be balanced by a single force, but can be balanced by a couple of opposite sign.

TRIANGLE THEOREM OF COUPLES

If three forces acting on a body be represented in magnitude, direction and line of action by the sides of triangle taken in order, then they are equivalent to a couple whose moment is represented by twice the area of triangle.



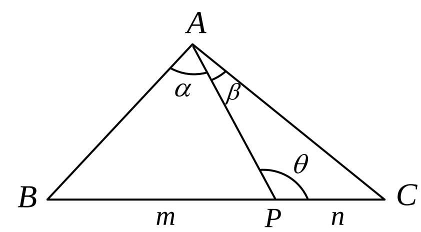
Consider the force P along AE, Q along CA and R along $AB.$ These forces are three concurrent forces acting at A and represented in magnitude and direction by the sides BC, CA and AB of ΔABC. So, by the triangle law of forces, they are in equilibrium.

The remaining two forces P along AD and P along BC form a couple, whose moment is, $m = P.AL = BC.AL$

Since, $\frac{1}{2}(BC.AL) =$ Area of the ΔABC

$\therefore$ Moment $= BC.AL = 2(\text{Area of } \Delta ABC).$

EQUILIBRIUM OF COPLANAR FORCES



1. If three forces keep a body in equilibrium, they must be coplanar.
2. If three forces acting in one plane upon a rigid body keep it in equilibrium, they must either meet in a point or be parallel.
3. When more than three forces acting on a rigid body, keep it in equilibrium, then it is not necessary that they meet at a point. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of their resolved parts in any two mutually perpendicular directions vanish separately, and if the algebraic sum of their moments about any point in their plane is zero.
   i.e., $X = 0, Y = 0, G = 0$ or $R = 0, G = 0.$
4. A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of the moments of the forces about each of three non-collinear points is zero.
5. Trigonometrical theorem : If P is any point on the base BC of ΔABC such that $BP : CP = m : n.$
   Then, (i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta,$
   where $\angle BAP = \alpha, \angle CAP = \beta.$
   (ii) $(n+m)\cot\theta = n\cot B - m\cot C.$

VARIGNON'S THEOREM OF MOMENTS

The algebraic sum of moments of any number of coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

FRICTION AND FORCE OF FRICTION

The property by virtue of which a resisting force is created between two rough bodies which prevents the sliding of one body over the other is called the friction and this force which always acts in the direction opposite to that in which the body has a tendency to slide or move is called forces of friction.

LIMITING FRICTION

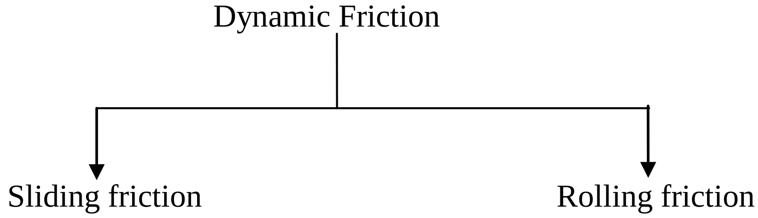
When one body is just on the point of sliding on another body, the force of friction called into play attains its maximum value and is called limiting friction and the equilibrium then is said to be limiting equilibrium.

STATIC FRICTION

When a body in contact with another body is in any position of equilibrium but not limiting equilibrium , then the friction exerted is called static friction. Thus, **static friction is less than the limiting friction.**

DYNAMIC FRICTION

When motion ensues by one body sliding on the other, the friction exerted between the bodies is called dynamic friction.



SLIDING FRICTION

If the body is sliding, the force of friction that comes into play is called sliding friction.

ROLLING FRICTION

If the body is rolling , the force of friction that comes into play is called rolling friction.

LAWS OF FRICTION

The following laws govern the different kinds of friction *i.e.* static, limiting and dynamic friction.

LAWS OF STATIC FRICTION

1. The direction of friction is opposite to the direction in which the body tends to move.
2. The magnitude of the force of friction is just sufficient to prevent the body from moving.

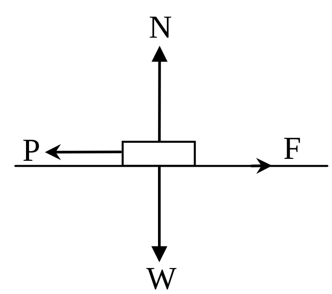
LAWS OF LIMITING FRICTION

1. Limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion.
2. The magnitude of limiting friction at the point of contact between two bodies bears a constant ratio to the normal reaction at the point.
3. The constant ratio depends entirely on the nature of the material of which the surfaces in contact are composed of and is independent of their extent and shape.

LAWS OF DYNAMIC FRICTION

1. The direction of dynamic friction is opposite to that in which the body is moving.
2. The magnitude of dynamic friction bears a constant ratio to the normal reaction on the body but this ratio is slightly less than the coefficient of friction in the case of limiting friction.
3. The dynamic friction is independent of the velocity of motion.

COEFFICIENT OF FRICTION



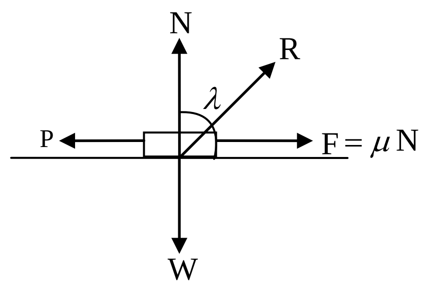
When a rough body is on the verge of sliding on another, the friction exerted bears a constant ratio to the normal reaction. This ratio of the limiting friction to the normal reaction is called the coefficient of friction. It is usually denoted by μ.

If F be the limiting friction and N, the normal reaction between the two bodies, then for the equilibrium to be limiting , we have

$$\frac{F}{N} = \mu \text{ or } = F = \mu N.$$

As friction is maximum when the equilibrium is limiting , μN is the maximum value of friction.

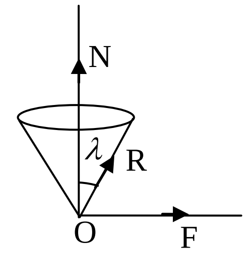
ANGLE OF FRICTION



When one body, placed on another body is in limiting equilibrium, the friction exerted is the limiting friction. In this case , the angle which the resultant of the force of friction and the normal reaction makes with the normal reaction at the point of contact is called the angle of friction and is usually denoted by λ.

Now, N and $\mu N(=F)$ being the resolved parts of R, we have $R\cos\lambda = N$ and $R\sin\lambda = \mu N \Rightarrow \tan\lambda = \mu$ Hence, the coefficient of friction is equal to the tangent of the angle of friction.

CONE OF FRICTION



The right cone described with its vertex at the point of contact of two rough bodies and having the common normal at the point of contact as axis and the angle of friction as the semi–vertical angle , is called the cone of fraction.

LEAST FORCE ON HORIZONTAL PLANE

The least force required to pull a body of weight W on the rough horizontal plane is W $\sin\lambda$.

LEAST FORCE ON INCLINED PLANE

Let α be the inclination of rough inclined plane to the horizontal and λ, the angle of friction.

1. If $\alpha = \lambda$, then the body is in limiting equilibrium and is just on the point of moving downwards.
2. If $\alpha < \lambda$, then the least force required to pull a body of weight W down the plane is $W \sin(\lambda - \alpha)$.
3. If $\alpha > \lambda$, then the body cannot rest on the plane under its own weight and reaction of the plane. So, the question of finding the least force does not arise.

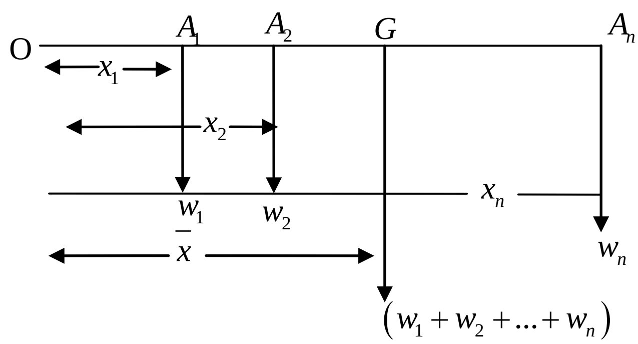
Note : The least force required to pull a body of weight W up an inclined rough plane is W $\sin(\alpha + \lambda)$.

CENTRE OF GRAVITY

The centre of gravity of a body or system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes.

CENTRE OF GRAVITY OF A NUMBER OF PARTICLES ARRANGED IN A STRAIGHT LINE

If n particles of weights $w_1, w_2, w_3, ..., w_n$ be placed at points $A_1, A_2, A_3, ..., A_n$ on the straight line OA_n such that the distance of these points from O are $x_1, x_2, ..., x_n$ respectively, Then, the distance of their centre of gravity G (say) from O is given by



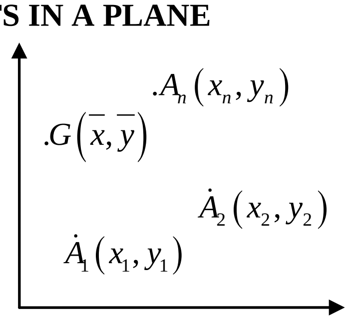
$$OG = \bar{x} = \frac{\sum w_i x_i}{\sum w_i}.$$

CENTRE OF GRAVITY OF A NUMBER OF WEIGHTS PLACED AT POINTS IN A PLANE

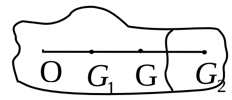
If $w_1, w_2, ..., w_n$ be the weights of the particles placed at the points $A_1(x_1, y_1), A_2(x_2, y_2) ..., A_n(x_n, y_n)$ respectively,

then the centre of gravity $G(\bar{x}, \bar{y})$ of these particles is given by

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i}, \bar{y} = \frac{\sum w_i y_i}{\sum w_i}.$$



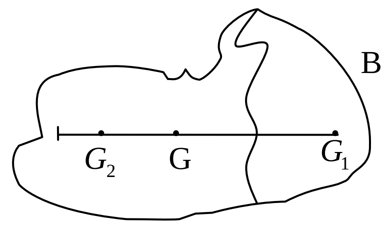
CENTRE OF GRAVITY OF COMPOUND BODY



Let G_1, G_2, be the centres of gravity of the two parts of a body and let w_1, w_2 be their weights. Let G be the centre of gravity of the whole body .Then at G, acts the whole weight $(w_1 + w_2)$ of the body. Join G_1G_2; then G must lie on G_1G_2.

Let O be any fixed point on G_1G_2. Let $OG_1 = x_1$, $OG_2 = x_2$ and $OG = \overline{x}$.Taking moments about O, we have $(w_1 + w_2)\overline{x} = w_1x_1 + w_2x_2$ $\therefore \overline{x} = \dfrac{w_1x_1 + w_2x_2}{w_1 + w_2}$

CENTRE OF GRAVITY OF THE RAMAINDER



Let w be the weight of the whole body .Let a part B of the body of weight w_1 be removed so that a part A of weight $w - w_1$ is left behind .Let G be the centre of gravity of whole body and G_1, the C.G. of portion B which is removed. Let G_2 be the C .G. of the remaining portion A. Let O be a point on G_1G_2 and let it be regarded as origin. Let $OG_1 = x_1, OG = x, OG_2 = x_2$ Taking moments about O, $(w - w_1)x_2 + w_1x_1 = wx \therefore x_2 = \dfrac{wx - w_1x_1}{w - w_1}$.

POSITION OF CENTRE OF GRAVITY IN SOME SPECIAL CASES:

1. **UNIFORM ROD :** At its mid point.
2. **PARALLELOGRAM, RECTANGLE OR SQUARE :** At the intersection of the diagonals.
3. **TRIANGULAR LAMINA :** At the centre.
4. **CIRCULAR ARC**

   At a distance $\dfrac{a\sin\alpha}{\alpha}$ form the centre on the symmetrical radius. Where a = radius and 2α = angle subtended by the arc C at the centre.
5. **SECTOR OF A CIRCLE**

   At a distance $\dfrac{2a}{3}\cdot\dfrac{\sin\alpha}{\alpha}$ form the centre on the symmetrical radius. Where a = radius and 2α = angle subtended at th centre.
6. **SEMI-CIRCULAR ARC**

   At a distance $\dfrac{4a}{3\pi}$ from the centre on the symmetrical radius, where α is the radius.
7. **HEMISPHERE**

   At a distance $\dfrac{3a}{8}$ from the centre on the symmetrical radius, where α is the radius.
8. **HEMISPHERICAL SHELL**

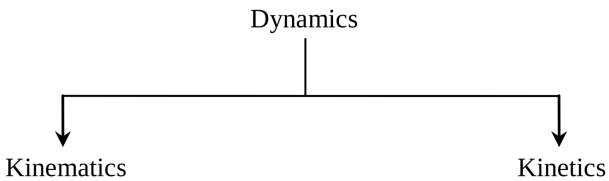
   At a distance $\dfrac{a}{2}$ from the centre on the symmetrical radius, where α is the radius.
9. **SOLID CONE**

   At a distance $\dfrac{h}{4}$ from the base on the axis, where h is the height of the cone.
10. **CONICAL SHELL**

    At a distance $\dfrac{h}{3}$ from the base on the axis, where h is the height of the cone.

Dynamics   Chapter 37

DYNAMICS
Dynamics is that branch of mechanics which deals with the motion of bodies under the action of forces



KINEMATICS
Kinematics is the study of geometry of motion without reference to the cause of motion. It deals with displacement, velocity, acceleration etc. and we establish relation between these and time without reference to the cause of motion.

KINETICS
Kinetics is the study of relationship between the forces and the resulting motion of bodies on which they act.

REST AND MOTION
A particle is said to be at rest if it does not change its position with respect to its surroundings and is said to be in motion if it changes its position with respect to its surroundings.

PATH
The straight line or the curve along which an object moves is called its path. If the path is a straight line, the object is said to have **rectilinear motion** and if the path is a curve the object is said to have **curvilinear motion.**

SPEED
The speed of a moving point is the rate at which it describes its path. Speed is a scalar quantity.

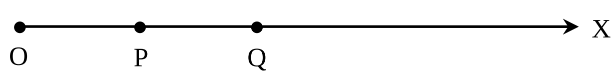
AVERAGE SPEED
Average speed of a moving object in a time interval is defined as the distance travelled by the object during that time interval divided by the time interval.

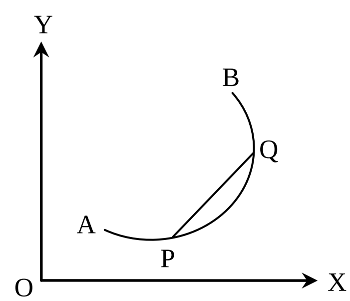
Thus, average speed in a time interval

$$= \frac{\text{distance travelled in the given time interval}}{\text{time interval}}$$

DISPLACEMENT
The displacement of a moving point is the distance covered by it in a definite direction.

This is the shortest distance between initial and fixed point and its direction is along the line from initial to final point.





A displacement has two fundamental characteristics-magnitude and direction. So it is a vector quantity.

VELOCITY
The rate of change of displacement of a moving particle is called its velocity.

Velocity has magnitude as well as direction. So it is a vector quantity.

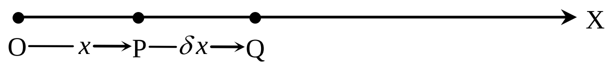
In M.K.S. system, its magnitude is measured in m/sec and in C.G.S. system, its magnitude is measured in cm/sec.

$$\text{Velocity of 1 Km/hr} = \frac{1\times100}{60\times60}\text{ m/sec.} = \frac{5}{18}\text{ m/sec}$$

$$\text{Velocity of 1 mile/hr} = \frac{22}{15} \text{ ft/sec.}$$

VELOCITY AT A POINT

Consider the motion of a particle along the straight line OX and let O be a point on it. Let P be the position of the particle at a time t and let $OP = x.$



Let Q be the position of the particle at time $(t+\delta t)$ and let $OQ = x + \delta x.$ The displacement of the particle in time $(t+\delta t) - t$ *.i.e.,* δt is PQ and $PQ = OQ - OP = (x+\delta x) - x = \delta x.$

$\therefore$ Displacement of the particle in time δt in $\delta x.$

Average velocity during the interval $\delta t = \dfrac{\delta x}{\delta t}.$

$\therefore$ Velocity at time t is given by, $v = \lim\limits_{\delta t \to 0} \dfrac{\delta x}{\delta t} = \dfrac{dx}{dt}.$

This is known as instantaneous velocity at $t.$

Thus, $v = \dfrac{dx}{dt}.$

UNIFORM VELOCITY

A particle is said to move with uniform velocity if it moves in a constant direction and covers equal distances in equal intervals of time, however small these intervals may be.

CONSTANT VELOCITY

A particle is said to move with constant velocity if it covers equal distances in equal intervals of time. Even if it is not moving in a constant direction.

AVERAGE VELOCITY

The average velocity of a particle in a given interval of time is given by the ratio of the total displacement undergone to the total time taken.

Thus, average velocity $= \dfrac{\text{Total displacement}}{\text{Total time}}$

ACCLERATION AND RETARDATION

The acceleration of a particle is the rate of increase in its velocity while retardation is the rate of decrease in velocity.

Thus, retardation is negative acceleration.

Unit of acceleration in C.G.S. system it is $cm/\sec^2$ and in M.K.S. system it is $m/\sec^2$.

Acceleration has magnitude as well as direction. So, it is a vector quantity.

EXPRESSION FOR ACCLERATION

If v is the velocity of a moving particle at a time $t,$ then the acceleration at time t is given by $a = \dfrac{dv}{dt}.$

OTHER EXPRESSIONS FOR ACCLERATION

1. $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2} \left[\because \dfrac{dx}{dt} = v\right]$

2. $a = \dfrac{dv}{dt} = \dfrac{dv}{dx} \cdot \dfrac{dx}{dt} \quad = v\dfrac{dv}{dx}\left[\because \dfrac{dx}{dt} = v\right]$

Thus, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dV}{dx}$.

Note :

(a) If $a = 0$, the particle is said to be moving with uniform velocity.

(b) If $a > 0$ then v increases with time.

(c) If $a < 0$ then v decreases with time.

(d) If v and a have the same sign then the speed of the particle is increasing.

(e) If v and a have the opposite signs then the speed of the particle is decreasing.

UNIFORM ACCLERATION

A particle is said to be moving with uniform acceleration, if equal changes in velocity take place in equal intervals of time, however small these interval may be.

EQUATIONS OF MOTION

If a particle moves along a straight line with initial velocity u and constant acceleration f, then the following relations are known as equations of motion :

(i) $v = u + ft$ (ii) $s = ut + \frac{1}{2}ft^2$ (iii) $v^2 - u^2 = 2fs$

where $'v'$ is the velocity after time t and s, the distance travelled in this time.

DISPLACEMENT IN THE n^{th} SECOND

The distance travelled by a particle in the n^{th} second of its motion in a straight line is given by

$$S_n = u + \frac{1}{2}f(2n-1)$$

MOTION UNDER GRAVITY

When a body is allowed to fall towards the earth, it will move vertically downwards with an acceleration which is always the same at the same place on the earth but varies slightly from place to place. This acceleration is called *acceleration due to gravity*. Its value in F.P.S. system is 32 $ft/\sec^2$, in C.G.S. system is 981 $cms/\sec^2$ and in M.K.S. system is 9.8 $m/\sec^2$. It is always denoted by g.

The acceleration due to gravity always acts vertically downwards. If the body moves downwards, then the effect of acceleration due to gravity is to increase its velocity. If the body moves upwards, then the effect of acceleration due to gravity is that of retardation *.i.e.*, the velocity of the body decreases. Hence, g is taken positive for the downwards motion and negative for the upward motion of a body.

DOWNWARD MOTION

If a body is projected vertically downwards from a point at a height h above the earth's surface with velocity u, the equation of its motion are

(i) $v = u + gt$ (ii) $h = ut + \frac{1}{2}gt^2$

(iii) $v^2 = u^2 + 2gh$ (iv) $h_n = u + \frac{1}{2}g(2n-1)$

Where h_n denotes the distance covered in the *n*th second

UPWARD MOTION

When a body is projected vertically upwards with initial velocity u, then it moves in a straight line with constant retardation g. So, the equations of motion in this case are :

(i) $v = u - gt$ (ii) $h = ut - \frac{1}{2}gt^2$

(iii) $v^2 = u^2 - 2gh$ and (iv) $h_n = u - \frac{1}{2}g(2n-1)$.

1. **GREATEST HEIGHT ATTAINTED**

At the highest point, velocity is zero and let H be the maximum height reached, therefore, from $v^2 = u^2 - 2gh,$ we get

$$0 = u^2 - 2gh \Rightarrow H = \frac{u^2}{2g}.$$

2. **TIME TO REACH THE GREATEST HEIGHT**

At the highest point, velocity is zero,

$$\therefore \quad 0 = u - gt \quad \Rightarrow \quad t = \frac{u}{g}$$

3. **TIME OF FLIGHT**

It is the total time taken by the particle to reach the greatest height and then return to the starting point.

When the particle returns to the starting point, $h = 0.$

we have, $0 = ut - \frac{1}{2}gt^2 \Rightarrow t = 0$ or $t = \frac{2u}{g},$ $t = 0$ corresponds to the instant when the body starts and $t = \frac{2u}{g}$ corresponds to the time when the body after attaining the greatest height reaches the starting point.

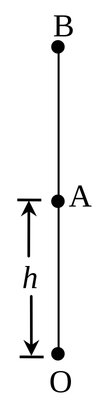
$\therefore$ $t = \frac{2u}{g}$ gives the required time of flight.

4. Time of descent = time of flight – time of ascent $= \frac{2u}{g} - \frac{u}{g} = \frac{u}{g}.$

$\therefore$ Time of ascent = time of descent and each is equal to half the time of flight.

5. **TIME TO REACH A GIVEN HEIGHT**

Let t be the time taken by the particle to reach a given height h



Then, $h = ut - \frac{1}{2}gt^2$ or $gt^2 - 2ut + 2h = 0$

This is a quadratic in t and gives two positive values of $t,$ say t_1 and t_2 If $t_1 < t_2$, then t_1 corresponds to the time when the particle attains the height h in the upward motion and t_2 is the time from the point of projection to the highest point and back to the given height $h, i.e., t_1$ is the time from O to A while t_2 is the time from O to B and B to $A.$

Also, $t_1 t_2 =$ product of roots $= \frac{2h}{g}.$

6. **VELOCITY AT A GIVEN HEIGHT**

Let v be the velocity of the particle at a given height $h.$ Then,

$$v^2 - u^2 = -2gh$$

$$\Rightarrow \quad v = \pm\sqrt{u^2 - 2gh}.$$

Thus, at a given height $h,$ the particle has two velocities which are equal in magnitude but opposite in sign. One of these represented the velocity in the upward direction and the other in the downward direction

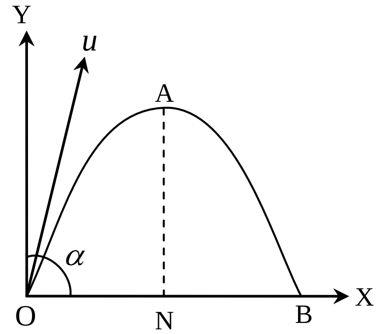
7. **VELOCITY ON REACHING THE GROUND**

Let v be the velocity of the body on reaching the ground. When the body is again at $O,$ displacement is zero, therefore, from $v^2 = u^2 - 2gh,$ we get, $v^2 = u^2 - 2g.0 \Rightarrow v^2 = u^2 \Rightarrow v = \pm u.$

But $v = u$ corresponds to the starting position, so we get $v = -u$ Thus, the magnitude of the velocity on reaching the ground is equal to the magnitude of velocity of projection and its direction is vertically downwards.

PROJECTILE MOTION

A particle projected in any direction is called a projectile.



TRAJECTORY

The path described by the particle is called its trajectory.

In figure, the curve OAB is the trajectory of the projectile.

VELOCITY OF PROJECTION

The velocity with which the particle is projected is called the velocity of projection.

In figure, u is the velocity of projection.

ANGLE OF PROJECTION

It is the angle which the direction of projection makes with the horizontal.

In figure, α is the angle of projection.

RANGE

The distance between the point of projection and the point where the projectile hits a given plane through the point of projection is called its range.

When the given plane is horizontal, it is called the *horizontal range*.

In figure, OB is the horizontal range of the projectile.

TIME OF FLIGHT

The time taken between the instant of projection and the instant when the projectile meets a fixed plane through the point of projection is called the time of flight. In figure, the time taken by the projectile in moving from O to B is the time of flight.

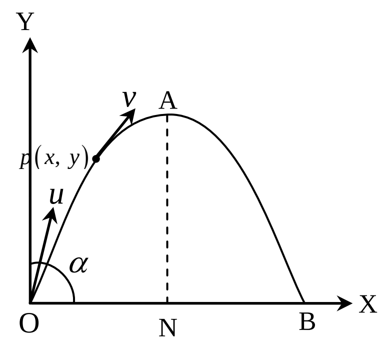
GREATEST HEIGHT

The maximum height reached by the particle above the point of projection during its motion is called the greatest height.

In figure, the distance AN is the greatest height.

SOME IMPORTANT RESULTS

Let a particle be projected from a point O with velocity u. Let the horizontal and vertical lines OX and OY in the plane of motion be taken as axes of reference. Let α be the angle of projection Let $P(x, y)$ be the position of the particle at any time t and v be the velocity of the particle at P. Let the direction of v makes an angle θ with the horizontal. Then



1. Horizontal distance covered in time t $= x = u\cos\alpha.t$

   Vertical distance covered in time t $= y = u\sin\alpha.t - \frac{1}{2}gt^2$

   Horizontal component of velocity at time t $= \frac{dx}{dt} = u\cos\alpha$

   Vertical component of velocity at time $t = \frac{dy}{dt} = u\sin\alpha - gt$

   Resultant velocity at time $t = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{u^2 - 2ugt.\sin\alpha + g^2t^2}$

   Direction of velocity at time t, $\theta = \tan^{-1}\left(\frac{u\sin\alpha - gt}{u\cos\alpha}\right)$

2. Equation of trajectory in

(i) parametric form $\begin{cases} x = u\cos\alpha\, t \\ y = u\sin\alpha\, t - \frac{1}{2}gt^2 \end{cases}$

(ii) General form $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$.

It is a parabola.

Its vertex is A $\left(\dfrac{u^2\sin\alpha.\cos\alpha}{g}, \dfrac{u^2\sin^2\alpha}{2g}\right)$.

Its focus is $S\left(\dfrac{u^2\sin^2 2\alpha}{2g}, \dfrac{-u^2\cos 2\alpha}{2g}\right)$

Its directrix is $y = \dfrac{u^2}{2g}$.

Latus rectum $= \dfrac{2u^2\cos^2\alpha}{g}$

$= \dfrac{2}{g}$ [Horizontal component of velocity $]^2$

It is the reciprocal of the numerical coefficient of x^2 in the equation of the trajectory.

3. Time of flight $= \dfrac{2u\sin\alpha}{g} = \dfrac{2}{g}$ [Vertical component of velocity]

4. Horizontal range, $R = \dfrac{u^2\sin 2\alpha}{g} = \dfrac{2u\cos\alpha.u\sin\alpha}{g}$

$= \dfrac{2}{g}$ [horizontal component of velocity] $\times$ [vertical component of velocity]

5. Greatest height attained $= \dfrac{u^2\sin^2\alpha}{2g}$

$= \dfrac{1}{2g}$ [vertical component of velocity $]^2$

6. Maximum horizontal range $= \dfrac{u^2}{g}$.

7. Angle of projection for maximum horizontal range $= \dfrac{\pi}{4}$.

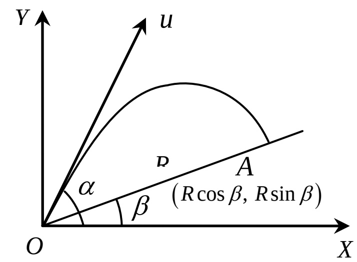
8. For a given range, there are two directions of projection which are complements of each other *.i.e.,* α and $90^0 - \alpha$.

9. Locus of the focus of trajectory is $x^2 + y^2 = \dfrac{u^2}{4g^2}$.

10. Locus of the vertex of trajectory is $x^2 + 4y^2 = \dfrac{2u^2 y}{g}$.

11. Velocity of the projectile at a height $h = \sqrt{u^2 - 2gh}$, Its direction is , $\theta = \tan^{-1}\left(\dfrac{\sqrt{u^2\sin^2\alpha - 2gh}}{4\cos\alpha}\right)$.

RANGE AND TIME OF FLIGHT ON AN INCLINED PLANE



Let OX and OY be the coordinate axes and OA be the inclined plane at an angle β to the horizon OX. Let a particle be projected from O with initial velocity u inclined at an angle α with the horizontal OX. The equation of the trajectory is $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$.

(i) Range and time of flight on an inclined plane with angle of inclination β are given by $R = \dfrac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$ and maximum range up the plane $= \dfrac{u^2}{g+(1+\sin\beta)}$, where $2\alpha-\beta = \dfrac{\pi}{2}$.

(ii) Time of flight $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$.

(iii) Range down the plane $= \dfrac{2u^2\cos\alpha\sin(\alpha+\beta)}{g\cos^2\beta}$

(iv) Maximum range down the plane $= \dfrac{u^2}{g(1-\sin\beta)}$, where $2\alpha+\beta = \dfrac{\pi}{2}$.

(v) Time of flight $= \dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$, down the plane.

(vi) Condition that the particle may strike the plane at right angles is $\cot\beta = 2\tan(\alpha-\beta)$.

NEWTON'S LAWS OF MOTION

FIRST LAW OF MOTION

Everybody continues in its state of rest or uniform motion in a straight line unless it is compelled by an external force to change that state.

This law asserts that a force is necessary to change

(i) the state of a body
(ii) the velocity of a body
(iii) the direction of motion.

INERTIA

Inertia is that property of a body by virtue of which it tries to continue its position of rest or of uniform motion in a straight line. Mass is the measure of inertia.

Newton's first law of motion is also, therefore, called the law of Inertia.

MOMENTUM OF A BODY

It is the quantity of motion possessed by a body and is equal to the product of its mass and velocity with which it moves.

Thus, if m be the mass of a body moving with velocity v, then its momentum is mv.

Momentum is a vector quantity. The direction of the momentum is the same as that of velocity.

The units of momentum are kg $m/\sec$ or gm $cm/\sec$.

SECOND LAW OF MOTION

The rate of change of momentum is directly proportional to the impressed force and it takes place in the direction along which the force acts.

RELATION BETWEEN FORCE, MASS AND ACCELERATION

If a force P acting on a body of mass m sets it in motion under acceleration f, the force P is given by $\mathrm{P} = mf$

.i.e., Force causing motion in absolute units

= mass × acceleration

This is called the **fundamental equation of Dynamics.**

Note : The equation $P = mf$ is valid only when the mass of the body in motion remains constant where as the force may be constant or variable. If the force is constant, the acceleration is uniform.

UNITS OF FORCE

1. **Dyne:** In C.G.S system, the unit of force is dyne. 1 dyne is the force which produces an acceleration of 1 $cm/\sec^2$ in a mass of 1g.
2. **Newton:** In MKS system, the unit of force is Newton. 1N is the force which produces an acceleration of 1 $m/\sec^2$ in a mass of 1 kg.
3. **Poundal:** In FPS system, the unit of force is poundal. 1 poundal is the force which produces an acceleration of 1 $ft/\sec^2$ in a mass of one pound.

Note : These units of force are called absolute units because their values are the same everywhere and do not depend upon the value of 'g' which varies from place to place on the Earth's surface.

GRAVITATIONAL UNITS

The units of force which depend on the value of 'g' are called gravitational units of force.

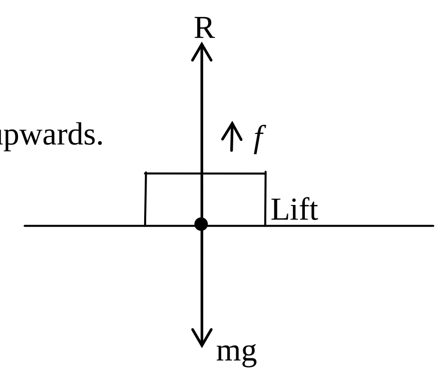
In CGS system, 1 gm. Wt. = g dynes =981 dynes.
In MKS system, 1 kg. wt.= g Newtons = 9.8 N.
In FPS system, 1 lb. wt. = g Poundal =32 poundals.

Note : While using the formula $P = mf$, P is always measured in absolute units *i.e.,* in poundals or dynes or Newtons.

THIRD LAW OF MOTION

To every action, there is an equal and opposite reaction. This law asserts that forces occur in pairs. When a book is placed on the table, the book presses the table with a certain force (which is action) and the table in turn presses the book with an equal but opposite force (which is reaction)

MOTION OF A LIFT MOVING UPWARDS



Suppose a body of mass m is carried by a lift upwards with an acceleration f .

The forces acting upon the body are:

(i) The normal reaction R of the plane of the lift acting vertically upwards.
(ii) The weight mg of the body acting vertically downwards.

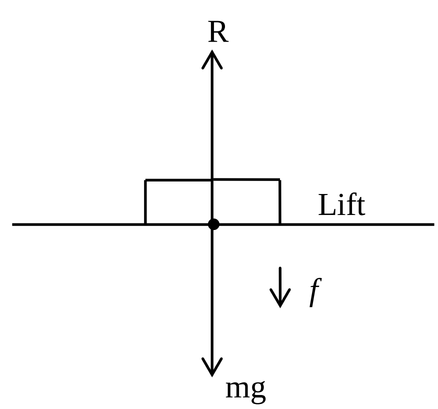
Since the lift is moving upwards, we have $R > mg$.

$\therefore$ Total upward forces acting on the body $= R - mg$.

$\therefore$ Equation of motion is $R - mg = mf$

$\therefore$ $R = m(g + f)$.

MOTION OF A LIFT MOVING DOWNWARDS



Let the lift be the moving vertically downwards.

Then $R < mg$

$\therefore$ Total downward force $= mg - R$

$\therefore$ Equation of motion is $mg - R = mf$

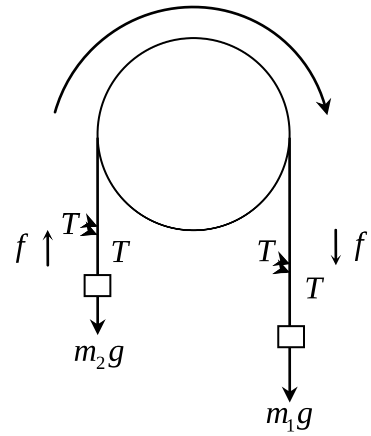
$\therefore$ $R = m(g - f)$.

Note : When $f = 0$, $R = mg$, *i.e.,* when the lift is at rest or moves with a constant velocity, the reaction is equal to the weight of the body.

MOTION OF TWO PARTICLES CONNECTED BY A STRING

1. **TWO PARTICLES HANGING VERTICALLY**

If two particles of masses m_1 and m_2 $(m_1 > m_2)$ are attached to the ends of a light inextensible string, which passes over a smooth fixed pulley, then for the system



Acceleration $f = \dfrac{(m_1 - m_2)g}{m_1 + m_2}$

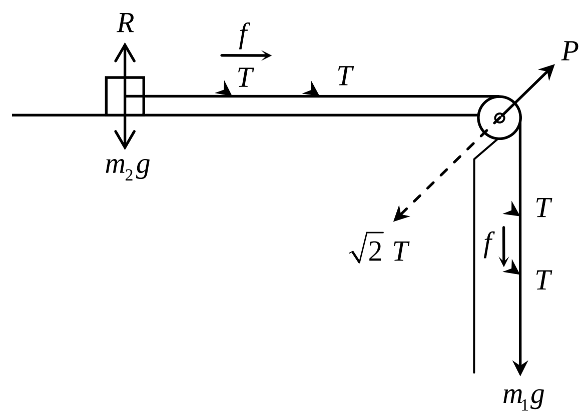
Tension in the string

$T = \dfrac{2m_1m_2g}{m_1 + m_2}$

Force on the pulley $= 2T = \dfrac{4m_1m_2g}{m_1 + m_2}$.

2. **ONE PARTICLE ON SMOOTH HORIZONTAL PLANE**

The particle of masses m_1 and m_2 $(m_1 > m_2)$ are connected by a light inextensible string. m_2 is placed on a smooth horizontal plane, the string passes over a light pulley at the edge of the plane and m_1 is hanging freely. Then for the system



Acceleration $f = \dfrac{m_1g}{m_1 + m_2}$

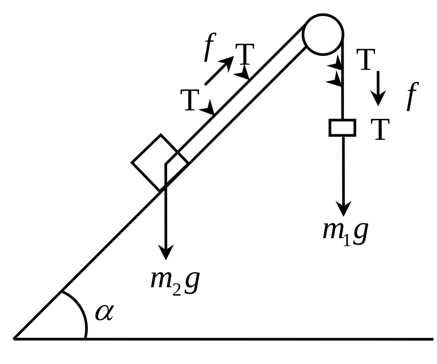
Tension of the string $T = \dfrac{m_1m_2g}{m_1 + m_2}$

Force on the pulley = $\sqrt{2}T = \dfrac{\sqrt{2}m_1m_2g}{m_1 + m_2}$.

3. **ONE PARTICLE ON AN INCLINED PLANE**

Two particles of masses m_1 and m_2 $(m_1 > m_2)$ are connected by a light inextensible string. m_2 is placed on a smooth plane inclined at an angle α to the horizontal, the string passes over a light pulley at the edge of the plane and m_1 is hanging freely

Then, For the system.



Acceleration $f = \dfrac{m_1 - m_2 \sin\alpha}{m_1 + m_2} g$

Tension in the string $T = \dfrac{m_1m_2(1 + \sin\alpha)g}{m_1 + m_2}$

Pressure on the pulley $= \sqrt{2(1 + \sin\alpha)}T \quad = \dfrac{\sqrt{2}m_1m_2(1 + \sin\alpha)^{3/2} g}{m_1 + m_2}$.

Elasticity : It is that property of bodies by virtue of which they can be compressed and after compression they recover or tend to recover their original shape. Bodies possessing this property are called elastic bodies.

Conservation of Linear momentum : If the external force acting on a system of particles (a body) is zero, then the net linear momentum of the system (a body) is conserved.

Thus in the absence of external forces.

Momentum before collision $=$ Momentum after collision

Mathematically $m_1.u_1 + m_2.u_2 = m_1.v_1 + m_2.v_2$.

where m_1 = mass of the first body, m_2 = mass of the second body,

u_1 = initial velocity of the first body, u_2 = initial velocity of the second body,

v_1 = final velocity of the first body, v_2 = final velocity of the second body.

Coefficient of Restitution : When the two bodies collide directly, then the ratio of relative velocity after collision to the relative velocity before collision is a fixed quantity. The quantity

$$e = -\frac{v_1 - v_2}{u_1 - u_2}$$

is called the coefficient of restitution. Here $u_1,\ u_2$ are initial velocity and $v_1,\ v_2$ are final velocity of two bodies. The value of e lies between 0 and 1. For a perfectly elastic collision $e = 1$ and for a perfectly inelastic collision $e = 0$.

Impact : When two bodies strike against each other, they are said to have an impact. It is of two kinds : Direct and Oblique.

Newton's experimental law of impact : It states that when two elastic bodies collide, their relative velocity along the common before impact and is in opposite direction.

If u_1 and u_2 be the velocities of the two bodies before impact along the common normal at their point of contact and v_1 and v_2 be their velocities after impact in the same direction, $v_1 - v_2 = -e\left(u_1 - u_2\right)$

Case I : If the two bodies move in direction shown in diagram given below, then

(a) $v_1 - v_2 = -e\left(u_1 - u_2\right)$

and (b) $m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$

Case II : If the direction of motion of two bodies before and after the impact are as shown below, then by the laws of direct impact, we have

(a) $v_1 - v_2 = -e\left[u_1 - \left(-u_2\right)\right]$ or $v_1 - v_2 = -e\left(u_1 + u_2\right)$

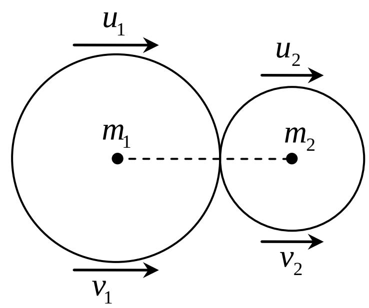
and (b) $m_1v_1 + m_2v_2 = m_1u_1 + m_2\left(-u_2\right)$ or $m_1v_1 + m_2v_2 = m_1u_1 - m_2u_2$

Case III : If the two bodies move in directions as shown below, then by the law of direct impact, we have

(a) $v_1 - \left(-v_2\right) = -e\left[-u_1 - \left(-u_2\right)\right]$ or $v_1 + v_2 = -e\left(u_2 - u_1\right)$

and (b) $m_1v_1 + m_2\left(-v_2\right) = m_1\left(-u_1\right) + m_2\left(-u_2\right)$ or $m_1v_1 - m_2v_2 = -\left(m_1u_1 + m_2u_2\right)$

Direct impact of two smooth spheres : Two smooth spheres of masses m_1 and m_2 moving along their line of centres with velocities u_1 and u_2 (measured in the same sense) impinge directly. To find their velocities immediately after impact, e being the coefficient of restitution between them.



Let v_1 and v_2 be the velocities of the two spheres immediately after impact, measured along their line of centres in the same direction in which u_1 and u_2 are measured. As the spheres are smooth, the impulsive action and reaction between them will be along the common normal at the point of contact. From the principle of conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad \ldots \text{(i)}$$

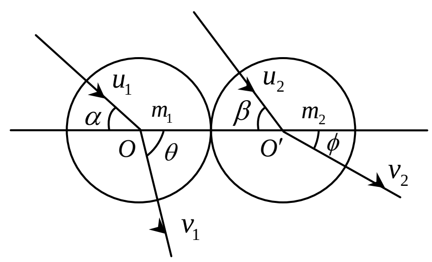
Also from Newton' experimental law of impact of two bodies, $v_2 - v_1 = e(u_1 - u_2)$ … (ii)

Multiplying (ii) by m_2 and subtracting from (i), we get

$$\Rightarrow \quad (m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2u_2(1+e)$$

$$\therefore \quad v_1 = \frac{(m_1 - em_2)u_1 + m_2u_2(1+e)}{m_1 + m_2} \qquad \ldots \text{(iii)}$$

Oblique impact of two spheres : A smooth sphere of mass m_1, impinges with a velocity u_1 obliquely on a smooth sphere of mass m_2 moving with a velocity u_2. If the direction of motion before impact make angles α and β respectively with the line joining the centres of the spheres, and if the coefficient of restitution be e, to find the velocity and directions of motion after impact.



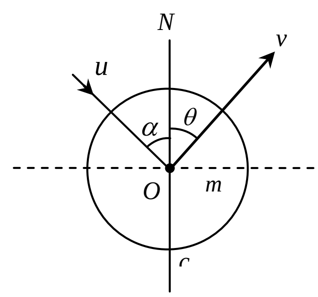
Let the velocities of the sphere after impact be v_1 and v_2 in directions inclined at angles θ and ϕ respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line joining the centres of the two balls and therefore, velocities in that direction are unaltered.

$$v_1 \sin\theta = u_1 \sin\alpha \qquad \ldots \text{(i)}$$

$$v_2 \sin\phi = u_2 \sin\beta \qquad \ldots \text{(ii)}$$

And by Newton's law, along the line of centres,

$$v_2 \cos\phi - v_1 \cos\theta = -e(u_2 \cos\beta - u_1 \cos\alpha) \qquad \ldots \text{(iii)}$$



Again, the only force acting on the spheres during the impact is along the line of centres. Hence the total momentum in that direction is unaltered.

$$\therefore \quad m_1v_1 \cos\theta + m_2v_2 \cos\phi = m_1u_1 \cos\alpha + m_2u_2 \cos\beta \qquad \ldots \text{(iv)}$$

The equations (i), (ii), (iii) and (iv) determine the unknown quantities.

Impact of a smooth sphere on a fixed smooth plane : Let a smooth sphere of mass m moving with velocity u in a direction making an angle α with the vertical strike a fixed smooth horizontal plane and let v be the velocity of the sphere at an angle θ to the vertical after impact.

Since, both the sphere and the plane are smooth, so there is no charge in velocity parallel to the horizontal plane.

$$\therefore \quad v \sin\theta = u \sin\alpha \qquad \ldots \text{(i)}$$

And by Newton's law, along the normal CN, velocity of separation $= e.$ (velocity of approach)

$\therefore \quad v\cos\theta - 0 = eu\cos\alpha$

$v\cos\theta = eu\cos\alpha$ … (ii)

Dividing (i) by (ii), we get $\cot\theta = e\cot\alpha$

Particular case : If $\alpha = 0$ then form (i), $v\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = 0$

$\therefore \quad \theta = 0$; $\quad \because \quad v \neq 0$ and from (ii) $v = eu$

Thus if a smooth sphere strikes a smooth horizontal plane normally, then it will rebound along the normal with velocity, e times the velocity of impact *i.e.* velocity of rebound $= e.$ (velocity before impact).

Rebounds of a particle on a smooth plane : If a smooth ball falls from a height h upon a fixed smooth horizontal plane, and if e is the coefficient of restitution, then whole time before the rebounding ends

$$= \sqrt{\left(\frac{2h}{g}\right)} \cdot \frac{1+e}{1-e}.$$

And the total distance described before finishing rebounding $= \dfrac{1+e^2}{1-e^2}h.$

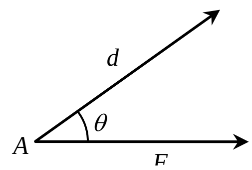
WORK, POWER AND ENERGY

1. **Work :** Work is said to be done by a force when its point of application undergoes a displacement. In other word, when a body is displaced due to the action of a force, then the forces are said to do work. Work is a scalar quantity.

   Work done $=$ force $\times$ displacement of body in the direction of force

   $W \quad = \vec{F}.\vec{d}$

   $W \quad = F\,d\cos\theta,$ where θ is the angle between F and d.



2. **Power :** The rate of doing work is called power. It is the amount of work that an agent is capable of doing in a unit time. 1 *watt* $=$ 10^7 *ergs per sec* $=1$ *joule per sec*., 1 $H.P. = 746$ *watt*.

3. **Energy :** Energy of a body is its capacity to do work it is of two kinds :

   (i) **Kinetic energy :** Kinetic energy is the capacity to do work by virtue of its motion and is measured by the work which the body can do against any force applied to stop it, before the velocity is destroyed.

   $$K.E. = \frac{1}{2}mv^2$$

   (ii) **Potential energy :** The potential energy of a body is the capacity to do work by virtue of its position of configuration. Potential energy of a particle having mass 'm' at height 'h' above the surface of the earth.

   $P.E. = mgh.$ Also, $K.E. + P.E. =$ Constant.

Mathematical Logic **Chapter 38**

INTRODUCTION

Logic was extensively developed in Greece. In the middle ages the treatises of Aristotle concerning logic were re-discovered. The axiomatic approach to logic was first proposed by **George Boole**. On this account logic relative to mathematics is sometimes called Boolean logic. It is also called Mathematical logic or more recently symbolic logic.

The word '*Logic*' means "*the science of reasoning*". It is the study and analysis of the nature of valid arguments.

STATEMENTS OR PROPOSITIONS

Propositions : A statement or a proposition is an assertive (or declarative) sentence which is either true or false but not both a true statement is called valid statement. If a statement is false, then it is called invalid statement.

Open statement : A declarative sentence containing variable (s) is an open statement if it becomes a statement when the variable (s) is (are) replaced by some definite value (s).

Truth Set : The set of all those values of the variable (s) in an open statement for which it becomes a true statement is called the truth set of the open statement.

Truth Value : The truth or falsity of a statement is called its truth value.

If a statement is true, then we say that its truth value is 'True' or 'T'. On the other hand the truth value of a false statement is 'False' or 'F'.

Logical variables : In the study of logic, statements are represented by lower case letters such as $p, q, r, s.$ These letters are called logical variables.

For example, the statement 'The sun is a star' may be represented or denoted by p and we write

p : The sun is a star

Similarly, we may denote the statement

$14 - 5 = -2.$

Quantifiers : The symbol $\forall$ (stands for 'for all') and $\exists$ (stands for 'there exists') are known as quantifiers.

In other word, quantifiers are symbols used to denote a group of words or a phrase.

The symbols $\forall$ and $\exists$ are known as existential quantifiers.

An open sentence used with quantifiers always becomes a statement.

Quantifies statements : The statements containing quantifiers are known as quantified statements.

$x^2 > 0 . \forall x \in R$ is a quantified statement. Its truth value is T.

USE OF VENN DIAGRAMS IN CHECKING TRUTH AND FALSITY OF STATEMENTS

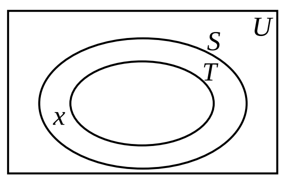
Venn diagrams are used to represent truth and falsity of statements or propositions. For this, let us consider the statement : "All teachers are scholars". Let us assume that this statement is true. To represent the truth of the above statement, we define the following sets

$U =$ the set of all human beings

$S =$ the set of all scholars

And $T =$ the set of all teachers

Clearly, $S \subset U$ and $T \subset U$



According to the above statement, if follows that $T \subset S$. Thus, the truth of the above statement can be represented by the Venn diagram shown in

Now, if we consider the statement : "All the scholars are teachers", It is evident from the Venn diagram that there is a scholar x who is not a teacher. Therefore, the above statement is false and its truth value is 'F'. Thus, we can also check the truth and falsity of other statements which are connected to a given statement.

TYPES OF STATEMENTS

In Mathematical logic there are two types of statements.

1. **Simple statements :** Any statement or proposition whose truth value does not explicity depend on another statement is said to be a simple statement.
   In other words, a statement is said to be simple if it cannot be broken down into simpler statements, that is, if it is not composed of simpler statements.
2. **Compound statements :** If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.

TRUTH TABLES

Definition : A table that shows the relationship between the truth value of a compound statement $S(p, q, r, ...)$ and the truth values of its sub-statement $p, q, r, ...$ etc, is called the truth table of statement S.

BASIC LOGICAL CONNECTIVES OR LOGICAL OPERATORS

Definition : The phrases or words which connect simple statements are called logical connectives or sentential connectives or simply connectives or logical operators.

List of some possible connectives, their symbols and the nature of the compound statement formed by them.

Connective	Symbol	Nature of the compound statement formed by using the connective
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
If …. then	$\Rightarrow$ or $\rightarrow$	Implication or conditional
If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	Equivalence or bi-conditional
not	$\sim$ or $\neg$	Negative

(i) **Conjunction :** Any two simple statements can be connected by the word "and" to form a compound statement called the conjunction of the original statements.
Symbolically if p and q are two simple statements, then $p \wedge q$ denotes the conjunction of p and q is read as "p and q".

(ii) **Disjunction or alternation :** Any two statements can be connected by the word "or" to form a compound statement called the disjunction of the original statements.
Symbolically, if p and q are two simple statements, then $p \vee q$ denotes the disjunction, of p and q and is read as "p or q".

(iii) **Negation :** The denial of a statement p is called its negation, written as $\sim p$.

Note :

Negation is called a connective although it does not combine two or more statements. In fact, it only modifies a statement.

(iv) **Implication or conditional statements :** Any two statements connected by the connective phrase "if …. then" give rise to a compound statement which is known as an implication or a conditional statement.
If p and q are two statements forming the implication 'if p then q', then we denote this implication by

$$"p \Rightarrow q" \text{ or } "p \rightarrow q".$$

In the implication $"p \Rightarrow q"$, p is the antecedent and q is the consequent.

Truth table for a conditional a statement

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(v) **Biconditional statement :** A statement is a biconditional statement if it is the conjunction of two conditional statements (implications) one converse to the other.

Thus, if p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called a biconditional statements or an equivalence and is denoted by $p \Leftrightarrow q$.

Thus, $p \Leftrightarrow q : (p \Rightarrow q) \wedge (q \Rightarrow p)$

Truth table for a biconditional statement : Since $p \Leftrightarrow q$ is the conjunction of $p \Rightarrow q$ and $q \Rightarrow p$. So, we have the following truth table for $p \Leftrightarrow q$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

LOGICAL EQUIVALENCE

Logically equivalent statement : Two compound statements $S_1(p, q, r, ...)$ and $S_2(p, q, r...)$ are said to be logically equivalent, or simply equivalent if they have the same truth values for all logically possibilities.

If statements $S_1(p, q, r...)$ and $S_2(p, q, r...)$ are logically equivalent, then we write

$$S_1(p, q, r, ...) \equiv S_2(p, q, r...)$$

It follows from the above definition that two statements S_1 and S_2 are logically equivalent if they have identical truth tables i.e., the entries in the last column of the truth tables are same.

NEGATION OF COMPOUND STATEMENTS

Writing the negation of compound statements having conjunction, disjunctions, implication, equivalence, etc, is not very simple.

1. **Negation of conjuntion :** If p and q are two statements, then $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$
2. **Negation of disjuntion :** If p and q are two statements, then $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$
3. **Negation of implication :** If p and q are two statements, then $\sim(p \Rightarrow q) \equiv (p \wedge \sim q)$
4. **Negation of biconditional statement or equivalence :**

   If p and q are two statements, then $\sim(p \Leftrightarrow q) \equiv (p \wedge \square\, q) \vee (q \wedge \sim p)$.

TAUTOLOGIES AND CONTRADICTIONS

Let $p, q, r, ...$ be statements, then any statement involving $p, q, r, ...$ and the logical connectives $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$ is called a statement pattern or a Well Formed Formula (WFF).

For example

(i) $p \vee q$

(ii) $p \Rightarrow q$

(iii) $\left(\left(p \wedge q\right) \vee r\right) \Rightarrow \left(s \wedge \sim s\right)$

(iv) $\left(p \Rightarrow q\right) \Leftrightarrow \left(\sim q \Rightarrow \sim p\right)$ etc.

are statement patterns. A statement is also a statement pattern.
Thus, we can define statement pattern as follows.

Statement pattern : A compound statement with the repetitive use of the logical connectives is called a statement pattern or a well-formed formula.

Tautology : A statement pattern is called a tautology, if it is always true, whatever may be the truth values of constitute statements.

A tautology is called a theorem or a logically valid statement pattern. A tautology, contains only T in the last column of its truth table.

Contradiction : A statement pattern is called a contradiction, if it is always false, whatever may the truth values of its constitute statements.

In the last column of the truth table of contradiction there is always F.

The negation of a tautology is a contradiction and vice versa.

Arguments : An argument is the assertion that statement S, follows from the other statement, $S_1, S_2 ....., S_n$.

We denote the argument by $\left(S_1, S_2, ..., S_n; S\right)$. The statement S is called the **conclusion** and the statement $S_1, S_2, ..., S_n$ are called **hypotheses** (or premises).

Validity of an argument. An argument consisting of the hypotheses $S_1, S_2 ...., S_n$ and conclusion S is said to be valid if S is true whenever all $S_1, S_2, ..., S_n$ are true.

Working rule to test an argument for validity:

1. Construct a truth table showing the truth values of all the hypotheses and the conclusion.
2. Find the rows (called critical rows) in which all the hypotheses are true. In case there exists no critical row, the argument is considered as **invalid.**
3. If in each critical row the conclusion is also true, then that argument is said to be valid otherwise the argument is invalid. (*i.e.*, if there is at least one critical row in which the conclusion is false, then the argument is invalid).

ALGEBRA OF STATEMENTS

The following are some laws of algebra of statements.

1. **Idempotent laws :** For any statement p, we have
   (a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$
2. **Commutative laws :** For any two statements p and q, we have
   (a) $p \vee q \equiv q \vee p$ (b) $p \wedge p \equiv q \wedge p$
3. **Association laws :** For any three statements p, q, r, we have
   (a) $\left(p \vee q\right) \vee r \equiv p \vee \left(q \vee r\right)$ (b) $\left(p \wedge q\right) \wedge r \equiv p \wedge \left(q \wedge r\right)$
4. **Distributive laws :** For any three statements p, q, r we have
   (a) $p \wedge \left(p \vee q\right) \equiv \left(p \wedge q\right) \vee \left(q \wedge r\right)$ (b) $p \vee \left(p \wedge q\right) \equiv \left(p \vee q\right) \wedge \left(q \vee r\right)$
5. **Demorgan's laws :** If p and q are two statements, then
   (a) $\sim \left(p \wedge q\right) \equiv \sim p \vee \sim q$ (b) $\sim \left(p \vee q\right) \equiv \sim p \wedge \sim q$
6. **Identity laws :** If t and c denote a tautology and a contradiction respectively, then for any statement p, we have
   (a) $p \wedge t \equiv p$ (b) $p \vee c \equiv p$ (c) $p \vee t \equiv t$ (d) $p \wedge c \equiv c$
7. **Complement laws :** For any statements p, we have
   (a) $p \vee \sim p = t$ (b) $p \wedge \sim p = c$ (c) $\sim t = c$ (d) $\sim c = t$

where t and c denote a tautology and a contradiction respectively.

8. **Law of contrapositive :** For any two statements p and q, the statement $(\sim q) \rightarrow (\sim p)$ is called the contrapositive of the statement $p \rightarrow q$.
9. **Involution laws :** For any statement p, we have $\sim(\sim p) \equiv p$

DUALITY

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) The connective $\wedge$ and $\vee$ are also called duals of each other.

(ii) If a compound statements contains the special variable t (tautology) or c (contradiction), then to obtain its dual we replace t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

(iii) Let $S(p, q)$ be a compound statement containing two sub-statements and $S^*(p, q)$ be its dual. Then,

(a) $\sim S(p, q) \equiv S^*(\sim p, \sim q)$

(b) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

(iv) The above result can be extended to the compound statements having finite number of sub-statements. Thus, if $S(p_1, p_2, ... p_n)$ is a compound statement containing n sub-statement $p_1, p_2, ..., p_n$ and $S^*(p_1 p_2, ..., p_n)$ is its dual. Then,

(a) $\sim S(p_1, p_2, ..., p_n) \equiv S^*(\sim p_1, \sim p_2, ..., \sim p_n)$

(b) $\sim S^*(p_1, p_2, ..., p_n) \equiv S(\sim p_1, \sim p_2, ..., \sim p_n)$.

Boolean Algebra

Chapter 39

BOOLEAN ALGEBRA

Definition : Boolean algebra is a tool for studying and applying mathematical logic which was originated by the English mathematician **George Boole**.

A non empty set B together with two operations denoted by '$\vee$' and '$\wedge$' is said to be a boolean algebra if the following axioms hold :

(i) For all $x, y \in B$

(a) $x \vee y \in B$ (Closure property for $\vee$)

(b) $x \wedge y \in B$ (Closure property for $\wedge$)

(ii) For all $x, y \in B$

(a) $x \vee y = y \vee x$ (Commutative law for $\vee$)

(b) $x \wedge y = y \wedge x$ (Commutative law for $\wedge$)

(iii) For all x, y and z in B,

(a) $(x \vee y) \vee z = x \vee (y \vee z)$ (Associative law of $\vee$)

(b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (Associative law of $\wedge$)

(iv) For all x, y and z in B,

(a) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (Distributive law of $\vee$ over $\wedge$)

(b) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (Distributive law of $\wedge$ over $\vee$)

(v) There exist elements denoted by 0 and 1 in B such that for all $x \in B$,

(a) $x \vee 0 = x$ (0 is identity for $\vee$)

(b) $x \wedge 1 = x$ (1 is identity for $\wedge$)

(vi) For each $x \in B$, there exists an element denoted by x', called the complement or negation of x in B such that

(a) $x \vee x' = 1$

(b) $x \wedge x' = 0$ (Complement laws)

PRINCIPLE OF DUALITY

The dual of any statement in a boolean algebra B is the statement obtained by interchanging the operation $\vee$ and $\wedge$, and simultaneously interchanging the elements 0 and 1 in the original statement.

In a boolean algebra, the zero element 0 and the unit element 1 are unique.

Let B be a boolean algebra. Then, for any x and y in B, we have

(a) $x \vee x = x (a') x \wedge x = x$

(b) $x \vee 1 = 1 (b') x \wedge 0 = 0$

(c) $x \vee (x \wedge y) = x$ $(c') x \wedge (x \vee y) = x$

(d) $0' = 1$ (d') $1' = 0$

(e) $(x')' = x$

(f) $(x \vee y)' = x' \wedge y'$ $(f)' (x \wedge y)' = x' \vee y'$

IMPORTANT POINTS

We sometimes designate a Boolean algebra by $(B, '\vee', '\wedge', '\,'', 0, 1)$ in order to emphasise its six parts; namely the set $B,$ the two binary operations '$\vee$' and '$\wedge$', the complement operation '′' and the two special elements 0 and 1. These special elements are called the zero element and the unit element. However, it may be noted that the symbols 0 and 1 do not necessarily represent the number zero and one.

Note :

For $P(A)$, the set of all subsets of a set $A,$ the operations $\cup$ and $\cap$ play the roles of '$\vee$' and '$\wedge$', A and ϕ play the role of 1 and 0, and complementation plays the role of '′'.

BOOLEAN EXPRESSION : Let $(B, '\vee', '\wedge', '\,'', 0, 1)$ be a Boolean algebra and $x_1, x_2, ..., x_n$ are in B. Then, Boolean expressions in $x_1, x_2, ..., x_n$ are defined recursively as follows:

(i) $0, 1, x_1, x_2, ..., x_n$ are all Boolean expressions.

(ii) If x and y are Boolean expression, then

(a) x' (b) $x \vee y\,(\text{or } x + y)$ (c) $x \wedge y\,(\text{or } x.y)$ (d) $x' \vee (y \wedge z)$

are also Boolean expressions. Here x', $x \vee y$, and $x \wedge y$ are called **monomials** and the expression $x' \vee (y \wedge z)$ is called a **polynomial**.

We denote a Boolean expression X in $x_1, x_2, ..., x_n$ by $X\,(x_1, x_2, ..., x_n)$.

BOOLEAN FUNCTIONS

Definition : Any expression like $x \wedge x',\ a \wedge b', [a \wedge (b \vee c')] \vee (a' \wedge b' \wedge c)$ consisting of combinations by $\vee$ and $\wedge$ of finite number of elements of a Boolean Algebra B is called a boolean function.

Let $B = \{a, b, c, ...\}$ be a boolean algebra by a constant we mean any symbol as 0 and 1, which represents a specified element of $B.$

By a variable we mean a symbol which represents a arbitrary element of B

If in the expression $x' \vee (y \wedge z)$ we replace $\vee$ by '$+$' and $\wedge$ by '.', we get $x' + y.z.$

SWITCHING CIRCUITS

One of the major practical application of Boolean algebra is to the switching systems (an electrical network consisting of switches) that involves two state devices. The simplest possible example of such a device is an ordinary **ON-OFF** switch.

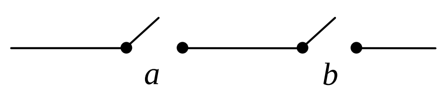
By a switch we mean a contact or a device in an electric circuit which lets (or does not let) the current to flow through the circuit. The switch can assume two states 'closed' or 'open' (**ON** or **OFF**). In the first case the current flows and in the second the current does not flow.

Symbols $a, b, c, p, q, r, x, y, z,$ ..... etc. will denote switches in a circuit.

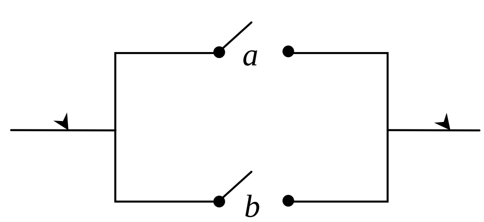
There are two basic ways in which switches are generally interconnected.

(i) Series (ii) Parallel

(i) **Series :** Two switches a, b are said to be connected 'in series' if the current can pass only when both are in closed state and the current does not flow if any one or both are open. The following diagram will show this circuit given by $a \wedge b$.



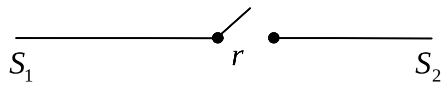
(ii) **Parallel :** Two switches a, b are said to be connected 'in parallel' if current flows when any one or both are closed, and current does not pass when both are open. The following diagram will represent this circuit given by $a \vee b.$



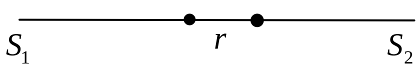
If two switches in a circuit be such that both are open (closed) simultaneously, we shall represent them by the same letter. Again if two switches be such that one is open and the other is closed, we represent them by a and a'.

The value of a **close** switch or when it is **on** is equal to 1 and when it is **open** or **off** is equal to 0.

An open switch r is indicated in the diagram as follows :

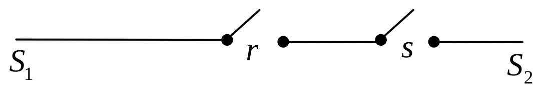


A closed switch r is indicated in the diagram as follows :



Boolean operations on switching circuits

(i) **Boolean Multiplication :** The two switches r and s in the series will perform the operation of Boolean multiplication.

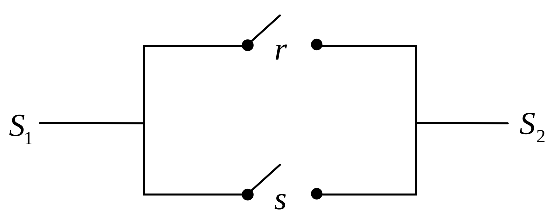


Clearly, the current will not pass from point S_1 to S_2 when either or both r, s are open . It will pass only when both are closed.

r	s	$r \wedge s$
1	1	1
1	0	0
0	1	0
0	0	0

The operation is true only in one of the four cases i.e. when both the switches are closed.

(ii) **Boolean Addition :** In the case of an operation of addition the two switches will be in the parallel series as shown below.



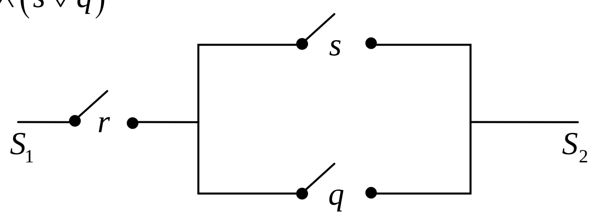
The circuit shows that the current will pass when either or both the switches are closed. It will not pass only when both are open.

r	s	$r \vee s$
1	1	1
1	0	1
0	1	1
0	0	0

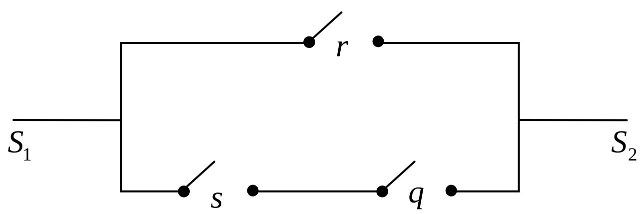
The operation is not true only in one of the four cases i.e., when both r and s are open.

(iii) **Circuits with composite operations**

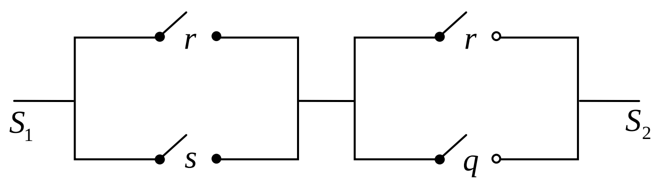
(a) Circuit showing : $r \wedge (s \vee q)$



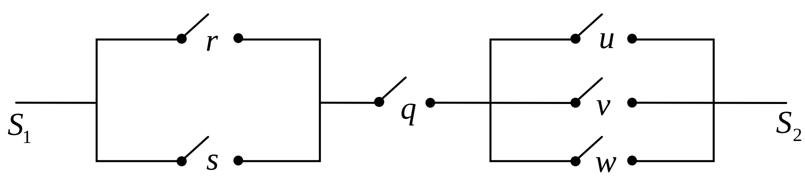
(b) Circuit showing $r \vee (s \wedge q)$



(c) Circuit showing $(r \vee s) \wedge (r \vee q)$



(iv) Circuit for : $(r \vee s) \wedge q \wedge (u \vee v \vee w)$

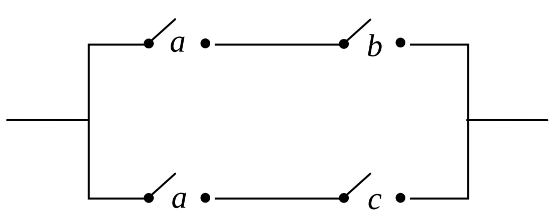


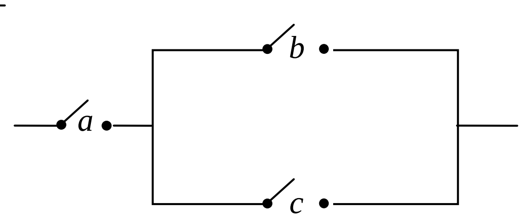
SIMPLIFICATION OF CIRCUITS

Simplification of a circuit would normally mean the least complicated circuit with minimum cost and best results. This would be governed by various factors like the cost of equipment, positioning and number of switches, types of material used etc. For us, simplification of circuits would mean lesser number of switches which we achieve by using different properties of Boolean algebra.

e.g., Consider the circuits given by $(a \wedge b) \vee (a \wedge c)$

This is represented by





Since $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$

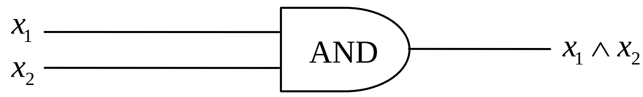
∴ The circuit could be simplified to

LOGIC GATES

(i) **AND :** It is the boolean function defined by

$$f\left(x_1,\ x_2\right) = x_1 \wedge x_2;\ x_1,\ x_2 \in \{0, 1\}.$$

It is shown in the figure given below.

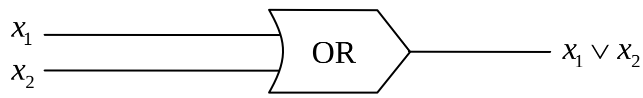


Input		Output
x_1	x_2	$x_1 \wedge x_2$
1	1	1
1	0	0
0	1	0
0	0	0

(ii) **OR :** It is the boolean function defined by

$$f\left(x_1,\ x_2\right) = x_1 \vee x_2;\ x_1, x_2 \in \{0, 1\}.$$

It is shown in the figure given below

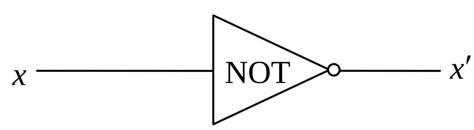


Input		Output
x_1	x_2	$x_1 \vee x_2$
1	1	1
1	0	1
0	1	1
0	0	0

(iii) **NOT :** It is the boolean function defined by

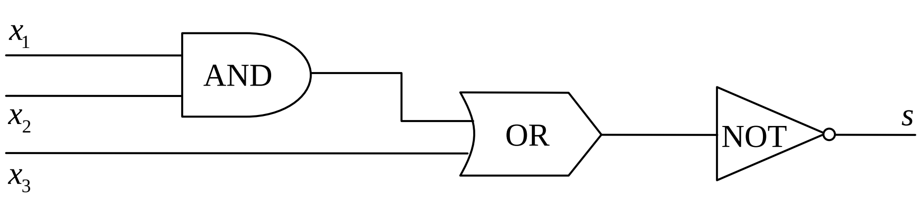
$$f\left(x\right) = x',\ x \in \{0, 1\}$$

It is shown in the figure given below

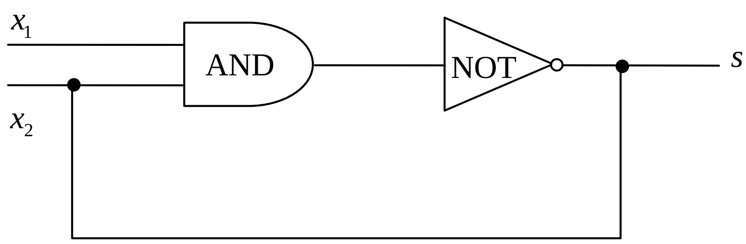


Input	Output
x	x'
1	0
0	1

COMBINATIONAL CIRCUIT



In the above figure, output s is uniquely defined for each combination of inputs x_1, x_2 and x_3. Such a circuit is called a combinatorial circuit or combinational circuit.



In the above figure, if $x_1 = 1, x_2 = 0,$ then the inputs to the AND gate are 1 and 0 and so the output of the AND gate is '0' (Minimum of 1 and 0). This is the input of NOT gate which gives the output $s = 1.$ But the diagram states that $x_2 = s$ i.e. $0 = 1$, a contradiction.

$\therefore$ The output s is not uniquely defined. This type of circuit is not a combinatorial circuit.

Two combinatorial circuits : Circuit having inputs $x_1, x_2, ... x_n$ and a single output are said to be combinatorial circuit if, the circuits receive the same input, they produce the same output i.e., if the input/output tables are identical.

Linear Programming — Chapter 40

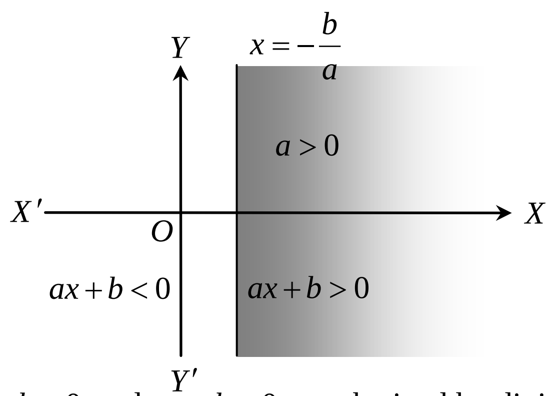
LINEAR PROGRAMMING

'Linear Programming' is a scientific tool to handle optimization problems.

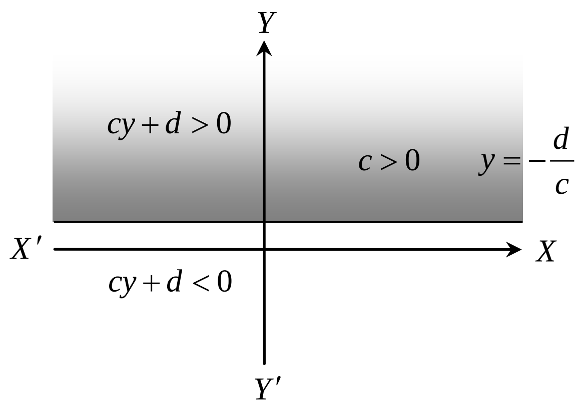
LINEAR INEQUATIONS

1. **Graph of linear inequations**

(i) **Linear inequation in one variable :** $ax+b>0$, $ax+b<0$, $cy+d>0$ etc. are called linear inequations in one variable. Graph of these inequations can be drawn as follows :

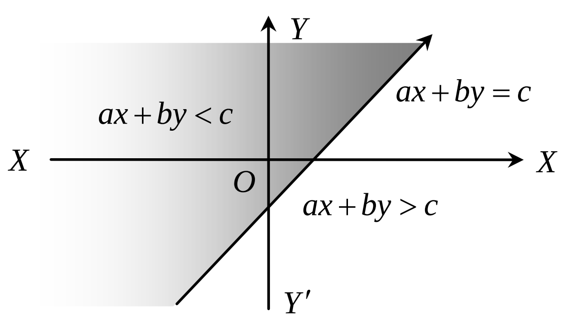


The graph of $ax+b>0$ and $ax+b<0$ are obtained by dividing xy- plane in two semi-planes by the line $x=-\frac{b}{a}$ (which is parallel to y-axis). Similarly for $cy+d>0$ and $cy+d<0$.



(ii) **Linear Inequation in two variables :** General form of these inequations are $ax+by>c$, $ax+by<c$. If any ordered pair (x_1, y_1) satisfies an inequation, then it is said to be a solution of the inequation.

The graph of these inequations is given below $(\text{for } c>0)$:



Working Rule : To draw the graph of an inequation, following procedure is followed.

(i) Write the equation $ax+by=c$ in place of $ax+by<c$ and $ax+by>c$.

(ii) Make a table for the solutions of $ax+by=c$.

(iii) Now draw a line with the help of these points. This is the graph of the line $ax+by=c$.

(iv) If the inequation is $>$ or $<$, then the points lying on this line is not considered and line is drawn dotted or discontinuous.

(v) If the inequation is $\geq$ or $\leq$, then the points lying on the line is considered and line is drawn bold or continuous.

(vi) This line divides the plane XOY in two region.

To Find the region that satisfies the inequation, we apply the following rules :

(a) Take an arbitrary point which will be in either region.

(b) If it satisfies the given inequation, then the required region will be the region in which the arbitrary point is located.

(c) If it does not satisfy the inequation, then the other region is the required region.

(d) Draw the lines in the required region or make it shaded.

2. **Simultaneous linear inequations in two variables :** Since the solution set of a system of simultaneous linear inequations is the set of all points in two dimensional space which satisfy all the inequations simultaneously. Therefore to find the solution set we find the region of the plane common to all the portions comprising the solution set of given inequations. In case there is no region common to all the solutions of the given inequations, we say that the solution set is **void** or **empty**.
3. **Feasible region :** The limited (bounded) region of the graph made by two inequations is called feasible region. All the points in feasible region constitute the solution of a system of inequations. The feasible solution of a L.P.P. belongs to only first quadrant. If feasible region is empty then there is no solution for the problem.

TERMS OF LINEAR PROGRAMMING

The term programming means planning and refers to a process of determining a particular program.

1. **Objective function :** The linear function which is to be optimized (maximized or minimized) is called objective function of the L.P.P.
2. **Constraints or Restrictions :** The conditions of the problem expressed as simultaneous equations or inequations are called constraints or restrictions.
3. **Non-negative constraints :** Variables applied in the objective function of a linear programming problem are always non-negative. The inequations which represent such constraints are called non-negative constraints.
4. **Basic variables :** The m variables associated with columns of the $m\times n$ non-singular matrix which may be different from zero, are called basic variables.
5. **Basic solution :** A solution in which the vectors associated to m variables are linear and the remaining $(n-m)$ variables are zero, is called a basic solution. A basic solution is called a degenerate basic solution, if at least one of the basic variables is zero and basic solution is called non-degenerate, if none of the basic variables is zero.
6. **Feasible solution :** The set of values of the variables which satisfies the set of constraints of linear programming problem (L.P.P.) is called a feasible solution of the L.P.P.
7. **Optimal solution :** A feasible solution for which the objective function is minimum or maximum is called optimal solution.
8. **Iso-profit line :** The line drawn in geometrical area of feasible region of L.P.P. for which the objective function (to be maximized) remains constant at all the points lying on the line, is called iso-profit line.
   If the objective function is to be minimized then these lines are called **iso-cost** lines.

9. **Convex set :** In linear programming problems feasible solution is generally a polygon in first quadrant. This polygon is convex. It means if two points of polygon are connected by a line, then the line must be inside the polygon. For example,

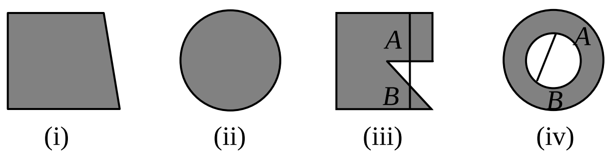


Fig. (i) and (ii) are convex set while (iii) and (iv) are not convex set.

MATHEMATICAL FORMULATION OF A LINEAR PROGRAMMING PROBLEM

There are mainly four steps in the mathematical formulation of a linear programming problem, as mathematical model. We will discuss formulation of those problems which involve only two variables.

1. Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.
2. Identify the set of constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the given conditions.
3. Identify the objective function and express it as a linear function of decision variables. It may take the form of maximizing profit or production or minimizing cost.
4. Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

GRAPHICAL SOLUTION OF TWO VARIABLE LINEAR PROGRAMMING PROBLEM

There are two techniques of solving an L.P.P. by graphical method. These are :

(1) Corner point method (2) Iso-profit or Iso-cost method

1. **Corner point method**

   Working Rule :

   (i) Formulate mathematically the L.P.P.
   (ii) Draw graph for every constraint.
   (iii) Find the feasible solution region.
   (iv) Find the coordinates of the vertices of feasible solution region.
   (v) Calculate the value of objective function at these vertices.
   (vi) Optimal value (minimum or maximum) is the required solution.
   (vii) If there is no possibility to determine the point at which the suitable solution found, then the solution of problem is unbounded.
   (viii) If feasible region is empty, then there is no solution for the problem.
   (ix) Nearer to the origin, the objective function is minimum and that of further from the origin, the objective function is maximum.

2. **Iso-profit or Iso-cost method :**

   Working Rule :

   (i) Find the feasible region of the L.P.P.
   (ii) Assign a constant value Z_1 to Z and draw the corresponding line of the objective function.
   (iii) Assign another value Z_2 to Z and draw the corresponding line of the objective function.
   (iv) If $Z_1 < Z_2, \left(Z_1 > Z_2\right)$, then in case of maximization (minimization) move the line P_1Q_1 corresponding to Z_1 to the line P_2Q_2 corresponding to Z_2 parallel to itself as far as possible, until the farthest point within the feasible region is touched by this line. The coordinates of the point give maximum (minimum) value of the objective function.
   (v) The problem with more equations/inequations can be handled easily by this method.

(vi) In case of unbounded region, it either finds an optimal solution or declares an unbounded solution. Unbounded solutions are not considered optimal solution.

TO FIND THE VERTICES OF SIMPLE FEASIBLE REGION WITHOUT DRAWING A GRAPH

1. **Bounded region :** The region surrounded by the inequations $ax+by \le m$ and $cx+dy \le n$ in first quadrant is called bounded region. It is of the form of triangle, quadrilateral or polygon. Change these inequations into equations, then by putting $x=0$ and $y=0,$ we get the solution. Also by solving the equations we get the vertices of bounded region.

   The maximum value of objective function lies at one of the vertex in bounded region.

2. **Unbounded region :** The region surrounded by the inequations $ax+by \ge m$ and $cx+dy \ge n$ in first quadrant, is called unbounded region.

   Change the inequation in equations and solve for $x=0$ and $y=0.$ Thus we get the vertices of feasible region.

The minimum value of objective function lies at one of the vertex in unbounded region but there is no existence of maximum value.

Hyperbolic Functions Chapter 41

Definition

We know that parametric co-ordinates of any point on the unit circle $x^2 + y^2 = 1$ is $(\cos\theta, \ \sin\theta)$; so these function are called circular functions and co-ordinates of any point on unit hyperbola $x^2 - y^2 = 1$ is $\left(\frac{e^{\theta}+e^{-\theta}}{2}, \frac{e^{\theta}-e^{-\theta}}{2}\right)$ i.e., $(\cosh\theta, \sinh\theta)$. It means that the relation which exists amongst $\cos\theta, \sin\theta$ and unit circle, that relation also exist amongst $\cosh\theta, \sinh\theta$ and unit hyperbola. Because of this reason these functions are called as Hyperbolic functions.

For any (real or complex) variable quantity *x*,

1. $\sinh x = \frac{e^x - e^{-x}}{2}$, [Read as 'hyperbolic sine *x*']
2. $\cosh x = \frac{e^x + e^{-x}}{2}$, [Read as 'hyperbolic cosine *x*']
3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
6. $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Note : $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

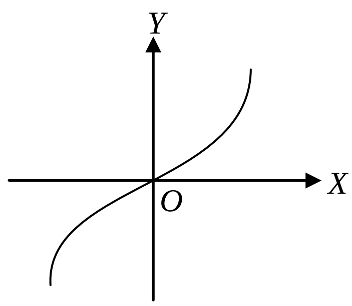
Domain and range of hyperbolic functions

Let *x* is any real number

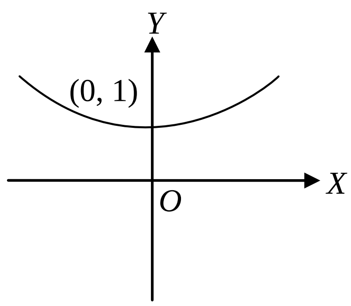
Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1, \infty)$
$\tanh x$	R	$(-1, 1)$
$\coth x$	$R - \{0\}$	$R - [-1, 1]$
$\operatorname{sech} x$	R	$(0, 1]$
$\operatorname{cosech} x$	$R - \{0\}$	$R - \{0\}$

Graph of real hyperbolic functions

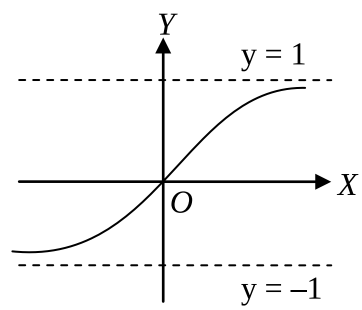
1. $\sinh x$



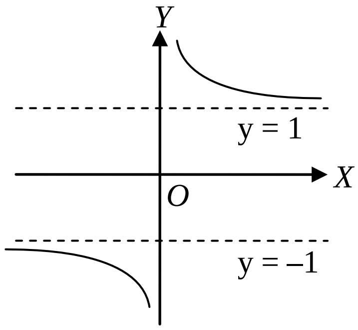
2. $\cosh x$



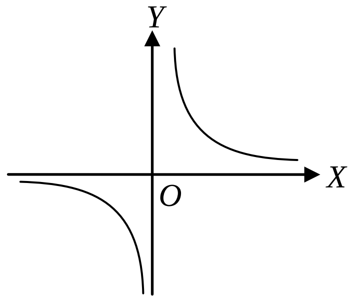
3. $\tanh x$



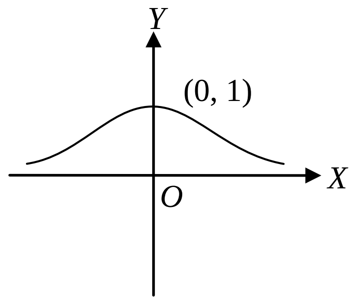
4. coth x



5. cosech x



6. sech x



Formulae for hyperbolic functions

The following formulae can easily be established directly from above definitions

1. **Reciprocal formulae**

(i) $\operatorname{cosech} x = \dfrac{1}{\sinh x}$ (ii) $\operatorname{sech} x = \dfrac{1}{\cosh x}$

(iii) $\coth x = \dfrac{1}{\tanh x}$ (iv) $\tanh x = \dfrac{\sinh x}{\cosh x}$

(v) $\coth x = \dfrac{\cosh x}{\sinh x}$

2. **Square formulae**

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $\operatorname{sech}^2 x + \tanh^2 x = 1$

(iii) $\coth^2 x - \operatorname{cosech}^2 x = 1$ (iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

3. **Sum and difference formulae**

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

4. **Formulae to transform the product into sum or difference**

(i) $\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$

(ii) $\sinh x - \sinh y = 2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(iii) $\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$

(iv) $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$

(v) $2\sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$

(vi) $2\cosh x \sinh y = \sinh(x+y) - \sinh(x-y)$

(vii) $2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$

(viii) $2\sinh x \sinh y = \cosh(x+y) - \cosh(x-y)$

5. **Trigonometric ratio of multiple of an angle**

(i) $\sinh 2x = 2\sinh x\cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 \quad = 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$ (iv) $2\sinh^2 x = \cosh 2x - 1$

(v) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$ (vi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(vii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (viii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

6. (i) $\cosh x + \sinh x = e^x$ (ii) $\cosh x - \sinh x = e^{-x}$

(iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Transformation of hyperbolic functions

Since, $\cosh^2 x - \sinh^2 x = 1$

$$\Rightarrow \quad \sinh x = \pm\sqrt{\cosh^2 x - 1} \quad \Rightarrow \quad \sinh x = \pm\frac{\sqrt{1-\text{sech}^2 x}}{\text{sech } x}$$

$$\Rightarrow \quad \sinh x = \pm\frac{\tanh x}{\sqrt{1-\tanh^2 x}} \quad \Rightarrow \quad \sinh x = \pm\frac{1}{\sqrt{\coth^2 x - 1}}$$

Also, $\sinh x = \dfrac{1}{\text{cosech } x}$

In a similar manner we can express $\cosh x, \tanh x, \coth x, \ldots\ldots\ldots.$ in terms of other hyperbolic functions.

Expansion of hyperbolic functions

1. $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots..$

2. $\cosh x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots..$

3. $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} - \dfrac{17}{315}x^7 + \ldots..$

The expansion of $\coth x$, cosech x does not exist because $\coth(0) = \infty$, $\text{cosech}(0) = \infty$.

Relation between hyperbolic, circular functions

1. $\sin(ix) = i\sinh x$
$\sinh(ix) = i\sin x$
$\sinh x = -i\sin(ix)$
$\sin x = -i\sinh(ix)$

2. $\cos(ix) = \cosh x$
$\cosh(ix) = \cos x$
$\cosh x = \cos(ix)$
$\cos x = \cosh(ix)$

3. $\tan(ix) = i\tanh x$
$\tanh(ix) = i\tan x$
$\tanh x = -i\tan(ix)$
$\tan x = -i\tanh(ix)$

4. $\cot(ix) = -i\coth$
$\coth(ix) = -i\cot x$
$\coth x = i\cot(ix)$
$\cot x = i\cot(ix)$

5. $\sec(ix) = \text{sech } x$

6. $\text{cosec}(ix) = i\text{cosech x}$

$\sec h(ix) = \sec x$ $\qquad$ $\text{cosech}(ix) = -i\,\text{cosec}\,x$

$\text{sech}\ x = \sec(ix)$ $\qquad$ $\text{cosech}\ x = i\,\text{cosec}(ix)$

$\sec x = \text{sech}(ix)$ $\qquad$ $\text{cosec}\,x = i\,\text{cosech}(ix)$

Period of hyperbolic functions

If for any function $f(x)$, $f(x+T) = f(x)$, then $f(x)$ is called the **Periodic function** and least positive value T is called the fundamental **Period** of the function.

$\because$ $\quad \sinh x = \sinh(2\pi i + x)$; $\qquad \cosh x = \cosh(2\pi i + x)$

and $\tanh x = \tanh(\pi i + x)$

Therefore, the fundamental period of these functions are respectively $2\pi i, 2\pi i$ and πi. Also period of $\text{cosech}\,x$, $\text{sech}\,x$ and $\coth x$ are respectively $2\pi i$, $2\pi i$ and πi.

Note :

(a) Remember that if the period of $f(x)$ is T, then period of $f(nx)$ will be $\left(\frac{T}{|n|}\right)$.

(b) Hyperbolic function are neither periodic functions nor their curves are periodic but they show the algebraic properties of periodic functions and having imaginary period.

Inverse hyperbolic functions

If $\sinh y = x$, then y is called the inverse hyperbolic sine of x and it is written as $y = \sinh^{-1} x$. Similarly $\text{cosec}^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ etc. can be defined.

1. **Domain and range of Inverse hyperbolic function**

Function	**Domain**	**Range**
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	$[1, \infty)$	R
$\tanh^{-1} x$	$(-1, 1)$	R
$\coth^{-1} x$	$R - [-1, 1]$	$R - \{0\}$
$\text{sech}^{-1} x$	$(0, 1]$	R
$\text{cosech}^{-1} x$	$R - \{0\}$	$R - \{0\}$

2. **Relation between inverse hyperbolic function and inverse circular function**

Method : Let $\sinh^{-1} x = y$

$\Rightarrow \quad x = \sinh y = -i\sin(iy) \;\Rightarrow\; ix = \sin(iy) \;\Rightarrow\; iy = \sin^{-1}(ix)$

$\Rightarrow \quad y = -i\sin^{-1}(ix) \;\Rightarrow\; \sinh^{-1} x = -i\sin^{-1}(ix)$

Therefore, we get the following relations

(i) $\sinh^{-1} x = -i\sin^{-1}(ix)$ $\qquad$ (ii) $\cosh^{-1} x = -i\cos^{-1} x$

(iii) $\tanh^{-1} x = -i\tan^{-1}(ix)$ $\qquad$ (iv) $\text{sech}^{-1} x = -i\sec^{-1} x$

(v) $\text{cosech}^{-1} x = i\,\text{cosec}^{-1}(ix)$

3. **To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions.**

To express $\sinh^{-1} x$ in terms of the others

(i) Let $\sinh^{-1} x = y \;\Rightarrow x = \sinh y \;\Rightarrow \text{cosech}\, y = \dfrac{1}{x} \;\Rightarrow y = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\because \quad \cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$

$\because \quad y = \cosh^{-1}\sqrt{1+x^2} \;\Rightarrow \sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2}$

(iii) $\because \quad \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{\sinh y}{\sqrt{1+\sinh^2 y}} = \dfrac{x}{\sqrt{1+x^2}}$

$\therefore \quad y = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}} \;\Rightarrow \sinh^{-1} x = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}}$

(iv) $\because \quad \coth y = \dfrac{\sqrt{1+\sinh^2 y}}{\sinh y} = \dfrac{\sqrt{1+x^2}}{x}$

$\therefore \quad y = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x} \;\Rightarrow \sinh^{-1} x = \coth^{-1}\dfrac{\sqrt{1+x^2}}{x}$

(v) $\because \quad \text{sech}\; y = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$

$y = \text{sech}^{-1}\dfrac{1}{\sqrt{1+x^2}} \;\Rightarrow \sinh^{-1} x = \text{sech}^{-1}\dfrac{1}{\sqrt{1+x^2}}$

(vi) Also, $\sinh^{-1} x = \text{cosech}^{-1}\left(\dfrac{1}{x}\right)$

From the above, it is clear that $\coth^{-1} x = \tanh^{-1}\left(\dfrac{1}{x}\right)$

$\text{sech}^{-1} x = \cosh^{-1}\left(\dfrac{1}{x}\right);\; \text{cosech}^{-1} = \sinh^{-1}\left(\dfrac{1}{x}\right)$

Note : If x is real then all above six inverse functions are single valued.

4. Relation between inverse hyperbolic functions and logarithmic functions

(i) $\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right),\ (-\infty < x < \infty)$

(ii) $\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right),\ (x \geq 1)$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right),\ |\,x\,| < 1$

(iv) $\coth^{-1} x = \dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right),\ |\,x\,| > 1$

(v) $\text{sech}^{-1} x = \log\left(\dfrac{1+\sqrt{1-x^2}}{x}\right),\ 0 < x \leq 1$

(vi) $\text{cosec}^{-1} x = \log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right),\ (x \neq 0)$

Note : Formulae for values of $\text{cosech}^{-1} x, \text{sech}^{-1} x$ and $\coth^{-1} x$ may be obtained by replacing x by $\dfrac{1}{x}$ in the values of $\sinh^{-1} x, \cosh^{-1} x$ and $\tanh^{-1} x$ respectively.

Separation of inverse trigonometric and inverse hyperbolic functions

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the inverse sine of $(x+iy)$.

We can write it as, $\sin^{-1}(x+iy) = \alpha + i\beta$.

1. $$\cos^{-1}(x+iy)=\frac{1}{2}\cos^{-1}\left[\left(x^2+y^2\right)-\sqrt{\left(1-x^2+y^2\right)^2+4x^2y^2}\right]$$
$$+\frac{1}{2}\cosh^{-1}\left[\left(x^2+y^2\right)+\sqrt{\left(1-x^2+y^2\right)^2+4x^2y^2}\right]$$

2. $$\sin^{-1}(x+iy)=\frac{\pi}{2}-\cos^{-1}(x+iy)\ =\frac{\pi}{2}-\frac{1}{2}\cos^{-1}\left[\left(x^2+y^2\right)-\sqrt{\left(1-x^2+y^2\right)^2+4x^2y^2}\right]$$
$$-\frac{i}{2}\cosh^{-1}\left[\left(x^2+y^2\right)+\sqrt{\left(1-x^2+y^2\right)^2+4x^2y^2}\right]$$

3. $$\tan^{-1}(x+iy)=\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)+\frac{i}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)\ =\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)+\frac{i}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]$$

4. $$\sin^{-1}(\cos\theta+i\sin\theta)=\cos^{-1}\left(\sqrt{\sin\theta}\right)+i\sinh^{-1}\left(\sqrt{\sin\theta}\right)\ \text{ or }\ \cos^{-1}\left(\sqrt{\sin\theta}\right)+i\log\left(\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\right)$$

5. $$\cos^{-1}(\cos\theta+i\sin\theta)=\sin^{-1}\left(\sqrt{\sin\theta}\right)-i\sinh^{-1}\left(\sqrt{\sin\theta}\right)\ \text{ or }\ \sin^{-1}\left(\sqrt{\sin\theta}\right)-i\log\left(\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\right)$$

6. $$\tan^{-1}(\cos\theta+i\sin\theta)=\frac{\pi}{4}+\frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right),\ (\cos\theta)>0$$

and $$\tan^{-1}(\cos\theta+i\sin\theta)=\left(-\frac{\pi}{4}\right)+\frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right),\ (\cos\theta)<0$$

Since, each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the appropriate method.

Note : Both inverse circular and inverse hyperbolic functions are many valued.

Numerical Methods **Chapter 42**

INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. The graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate.

SIGNIFICANT DIGITS AND ROUNDING OFF OF NUMBERS

1. **Significant digits :** The significant digits in a number are determined by the following rules :
   (i) All non-zero digits in a number are significant.
   (ii) All zeros between two non-zero digits are significant.
   (iii) If a number having embedded decimal point ends with a non-zero or a sequences of zeros, then all these zeros are significant digits.
   (iv) All zeros preceding a non-zero digit are non-significant.
2. **Rounding off of numbers :** If a number is to be rounded off to n significant digits, then we follow the following rules :
   (i) Discard all digits to the right of the n^{th} digit.
   (ii) If the $(n+1)^{th}$ digit is greater than 5 or it is 5 followed by a non-zero digit, then n^{th} digit is increased by 1. If the $(n+1)^{th}$ digit is less than 5, then digit remains unchanged.
   (iii) If the $(n+1)^{th}$ digit is 5 and is followed by zero or zeros, then nth digit is increased by 1 if it is odd and it remains unchanged if it is even.

If a number is rounded off according to the rules, the maximum error due to rounding does not exceed the one half of the place value of the last retained digit in the number.

ROUND OFF ERROR

The difference between a numerical value X and its rounded value X_1 is called **round off error** is given by $E = X - X_1$.

TRUNCATION AND ERROR DUE TO TRUNCATION OF NUMBERS

Leaving out the extra digits that are not required in a number without rounding off, is called truncation or chopping off.

The difference between a numerical value X and its truncated value X_1 is called truncation error and is given by $E = X - X_1$.

The maximum error due to truncation of a number cannot exceed the place value of the last retained digit in the number.

The difference between the exact value of a number X and its approximate value X_1, obtained by rounding off or truncation, is known as absolute error.

The quantity $\frac{X - X_1}{X}$ is called the relative error and is denoted by E_R.

Thus $E_R = \frac{X - X_1}{X} = \frac{\Delta X}{X}$. This is a dimensionless quantity.

The quantity $\frac{\Delta X}{X} \times 100$ is known as percentage error and is denoted by E_p, i.e. $E_p = \frac{\Delta X}{X} \times 100$.

ALGEBRAIC AND TRANSCENDENTAL EQUATION

An equation of the form $f(x)=0$ is said to an algebraic or a transcendental equation according as $f(x)$ is a polynomial or a transcendental function respectively.

e.g. $ax^2+bx+c=0,\ ax^3+bx^2+cx+d=0$ etc., where $a, b, c, d \in Q$ are algebraic equations whereas $ae^x+b\sin x=0;\ a\log x+bx=3$ etc. are transcendental equations.

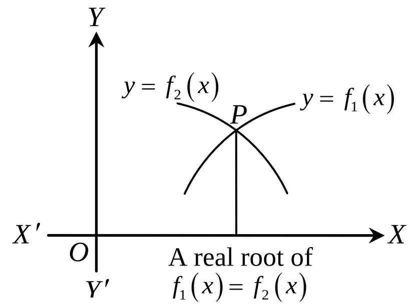
Note : A function which is not algebraic is called transcendental equations.

POSITION OF REAL ROOTS

By location of a real root of an equation, we mean finding an approximate value of the root graphically or otherwise.

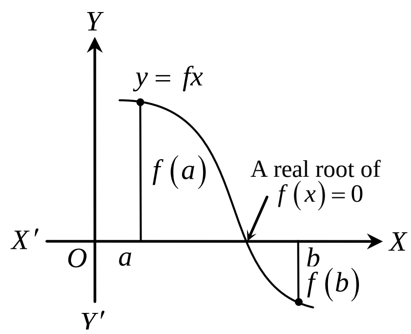
1. **Graphical Method :** It is often possible to write $f(x)=0$ in the form $f_1(x)=f_2(x)$ and then plot the graphs of the functions $y=f_1(x)$ and $y=f_2(x)$.

   The abscissae of the points of intersection of these two graphs are the real roots of $f(x)=0$.



2. **Location Theorem :** Let $y=f(x)$ be a real-valued, continuous function defined on $[a, b]$.

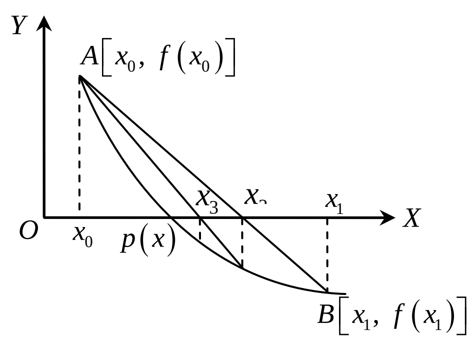
   If $f(a)$ and $f(b)$ have opposite signs i.e., $f(a)f(b)<0$, then the equation $f(x)=0$ has at least one real root between a and b.



SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS (ITERATIVE METHODS)

There are many numerical methods for solving algebraic and transcendental equations. Some of these methods are given below. After locating root of an equation, we successively approximate it to any desired degree of accuracy.

1. **Successive bisection method :** This method consists in locating the root of the equation $f(x)=0$ between a and b. If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs i.e., $f(a).f(b)<0$, then there is at least one root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root $x_1=\frac{1}{2}(a+b)$.



Working Rule

(i) Find $f(a)$ by the above formula.

(ii) Let $f(a)$ be negative and $f(b)$ be positive, then take $x_1=\dfrac{a+b}{2}$.

(iii) If $f(x_1) = 0$, then x_1 is the required root or otherwise if $f(x_1)$ is negative then root will be in (x_1, b) and if $f(x_1)$ is positive then root will be in (a, x_1).

(iv) Repeat it until you get the root nearest to the actual root.

2. **Method of false position or Regula-Falsi method :** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e., the graph of $y = f(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 and x_1, consequently $f(x_0)\,f(x_1) < 0$.

Equation of the chord joining the points

$$A[x_0,\ f(x_0)] \text{ and } B[x_1,\ f(x_1)] \text{ is } y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad \ldots \text{(i)}$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis $(y = 0)$ is given by $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$ … (ii)

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (ii), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (i) is known as the method of false position.

Working Rule

(i) Calculate $f(x_0)$ and $f(x_1)$, if these are of opposite sign then the root lies between x_0 and x_1.

(ii) Calculate x_2 by the above formula.

(iii) Now if $f(x_2) = 0$, then x_2 is the required root.

(iv) If $f(x_2)$ is negative, then the root lies in (x_2, x_1).

(v) If $f(x_2)$ is positive, then the root lies in (x_0, x_2).

(vi) Repeat it until you get the root nearest to the real root.

3. **Newton-Raphson method :** Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get $f(x_0) + hf'(x_0) = 0$

or $$h = -\frac{f(x_0)}{f'(x_0)} \qquad \ldots \text{(i)}$$

$\therefore$ A closer approximation to the root is given by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, {from (i)}

Similarly, starting with x_1, a still better approximation x_2 is given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

In general, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

which is known as the Newton-Raphson formula or Newton's iteration formula.

Working Rule

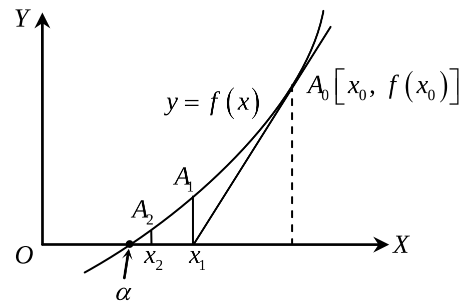
(i) Find $|f(a)|$ and $|f(b)|$. If $|f(a)| < |f(b)|$, then let $a = x_0$, otherwise $b = x_0$.

(ii) Calculate $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

(iii) x_1 is the required root, if $f(x_1) = 0$.

(iv) To find nearest to the real root, repeat it.

GEOMETRICAL INTERPRETATION



Let x_0 be a point near the root α of the equation $f(x) = 0$. Then the equation of the tangent at $A_0[x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, which is a first approximation to the root α.

If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.

NUMERICAL INTEGRATION

It is the process of computing the value of a definite integral when we are given a set of numerical values of the integrand $f(x)$ corresponding to some values of the independent variable x.

If $I = \int_a^b y.dx$. Then I represents the area of the region R under the curve $y = f(x)$ between the ordinates $x = a,\ x = b$ and the x-axis.

1. **Trapezoidal rule :** Let $y = f(x)$ be a function defined on $[a, b]$ which is divided into n equal sub-intervals each of width h so that $b - a = nh$.

Let the values of $f(x)$ for $(n+1)$ equidistant arguments

$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ ......,\ x_n = x_0 + nh = b$ be $y_0,\ y_1,\ y_2,\ ....y_n$ respectively.

Then $\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} y\,dx$

$$= h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + ... + y_{n-1})\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

This rule is known as **Trapezoidal rule.**

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2);....... ; (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y = f\left(x\right)$. The ordinate $x = x_0$ and $x = x_n$ and the x-axis, is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

2. **Simpson's one third rule :** Let $y = f\left(x\right)$ be a function defined on $\left[a, b\right]$ which is divided into n (an even number) equal parts each of width $h,$ so that $b - a = nh.$

Suppose the function $y = f\left(x\right)$ attains values $y_0, y_1, y_2, \ldots.. y_n$ at $n+1$ equidistant points $x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h, \ldots\ldots. x_n = x_0 + nh = b$ respectively. Then

$$\int_a^b f\left(x\right) dx = \int_{x_0}^{x_0+nh} y\, dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + y_5 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\right]$$

$=$ (one-third of the distance between two consecutive ordinates)

$\times$ [(sum of the extreme ordinates) $+4$ (sum of odd ordinates) $+2$ (sum of even ordinates)]

This formula is known as Simpson's one-third rule. Its geometric significance is that we replace the graph of the given function by $\frac{n}{2}$ arcs of second degree polynomials, or parabolas with vertical axes. It is to be note here that the interval $\left[a, b\right]$ is divided into an even number of subinterval of equal width.

Simpson's rule yield more accurate results than the trapezoidal rule. Small size of interval gives more accuracy.

Check your Intelligence — Chapter 43

CHECK YOUR INTELLIGENCE IN MATHEMATICS

1. Here I have proved the wrong statement that

$$\lim_{x\to 0} \cos\frac{1}{x} = 0$$

find the mistake in the following proof.

Proof : Let $f(x) = \begin{cases} x^2 \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Now let us apply Lagrange's theorem on this function in the interval $[0, x]$

Clearly is differentiable for any x by Lagrange's theorem

$x^2 \sin\frac{1}{x} = 2\xi \sin\frac{1}{\xi} - \cos\frac{1}{\xi}$

Hence $\cos\frac{1}{\xi} = 2\xi \sin\frac{1}{\xi} - x^2 \sin\frac{1}{x}$ where

As x tends to zero, ξ $(\because \xi \in (0, x))$ will also tend to zero therefore passing to the limit, we obtain $\lim_{\xi\to 0}\cos\frac{1}{\xi} = 0$.

2. Here I have proved the wrong statement that $\pi = 2\sqrt{2}$. Find the mistake in the following proof.

Proof : Consider the integral $I = \int_0^{\pi} xf(\sin x)\,dx,$ where $f(\sin x)$ is any function of sin x. In accordance with a standard treatment, make the substitution $x = \pi - x'$, and then drop dashes.

Thus $\quad I = \int_{\pi}^{0} (\pi - x) f\{\sin(\pi - x)\} d(-x) = \int_0^{\pi} (\pi - x) f(\sin x)\,dx.$

Hence $\quad 2\int_0^{\pi} xf(\sin x)\,dx = \pi\int_0^{\pi} f(\sin x)\,dx.$

Take, in particular,

$$f(u) = u\sin^{-1}(u), \text{ so that } f(\sin x) = \sin x . x = x\sin x.$$

Then the relation is

$$2\int_0^{\pi} x^2 \sin x dx = \pi\int_0^{\pi} x\sin x dx.$$

But $\quad \int_0^{\pi} x^2 \sin x dx = \pi^2 - 4, \ \int_0^{\pi} x\sin x dx = \pi.$

Hence $\quad 2(\pi^2 - 4) = \pi^2, \text{or } \pi^2 = 8.$

so that $\quad \pi = 2\sqrt{2}.$

3. Here I have proved the wrong statement that every angle is multiple of two right angles. Find the mistake in following proof.

Proof : Let θ be an angle (complex) satisfying the relation $\tan\theta = i$.

Then, if A is any angle,

$$\tan(A+\theta)=\frac{\tan A+\tan\theta}{1-\tan A\tan\theta}=\frac{\tan A+i}{1-i\tan A}=i=\tan\theta.$$

Thus, $\tan(A+\theta)=\tan\theta$, so that $A+\theta=n\pi+\theta$, or $A=n\pi$ for any angle A.

4. Here I have proved the wrong statement that $0=1$. Find the mistake in the following proof.

Proof : Consider the integral $I=\int\frac{dx}{x}$. Integrate by parts :

$$I=\int 1.(1/x)\,dx \quad = x(1/x)-\int x\left(-1/x^2\right)dx \quad =1+\int\frac{dx}{x} \quad =1+I \qquad \text{Hence } 0=1.$$

5. Here I have proved the wrong statement that $2=1$. Find the mistake in the following proof.

Proof : Let $f(x)$ be any given function.

Then $\int_1^2 f(x)\,dx=\int_0^2 f(x)\,dx-\int_0^1 f(x)\,dx.$

If we write $x=2y$ in the first integral on the right, then

$$\int_0^2 f(x)\,dx=2\int_0^1 f(2y)\,dy=2\int_0^1 f(2x)\,dx,$$

on renaming the variable. Suppose, in particular, that the function $f(x)$ is such that

$$f(2x)=\frac{1}{2}f(x)$$

for all values of x. Then

$$\int_1^2 f(x)\,dx=2\int_0^1\frac{1}{2}f(x)\,dx-\int_0^1 f(x)\,dx=0.$$

Now the relation $f(2x)=\frac{1}{2}f(x)$ is satisfied by the function $f(x)=\frac{1}{x}$

Hence $\int_1^2\frac{dx}{x}=0$, so that $\log 2=0$ or $2=1$.

6. Here I have proved the wrong statement that, if $f(\theta)$ is any function of θ, then $\int_0^\pi f(\theta)\cos\theta\,d\theta=0.$

Find the mistake in the following proof.

Proof : Substitute $\sin\theta=t$ so that $\cos\theta\,d\theta=dt$, and write $f\left\{\sin^{-1}t\right\}=g(t)$

The limits of integration are 0, 0 $\sin ce \sin 0=0$ and $\sin\pi=0$. Hence the integral is $\int_0^0 g(t)\,dt=0.$

Corollary : The special case when $f(\theta)=\cos\theta$ is of interest. Then the integral is

$$\int_0^\pi\cos^2\theta\,d\theta=\frac{1}{2}\int_0^\pi(1+\cos 2\theta)\,d\theta \quad =\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^\pi \quad =\frac{1}{2}\pi,\ \text{hence}\ \frac{1}{2}\pi=0.$$

7. Here I have proved the wrong statement that, $1=0$. Find the mistake in the following proof.

Proof : $S=1-1+1-1+1-1+\ldots$

Then, grouping in pairs,

$$S=(1-1)+(1-1)+(1-1)+\ldots \quad =0+0+0+\ldots \quad =0.$$

Also, grouping alternatively in pairs,

$$S=1-(1-1)-(1-1)-(1-1)-\ldots \quad =1-0-0-0\ldots=1.$$

Hence $1 = 0.$

8. Here I have proved the wrong statement that, –1 is positive. Find the mistake in the following proof.

Proof : Let $S = 1 + 2 + 4 + 8 + 16 + 32 + ...$

Then S is positive. Also, multiplying each side by 2,

$$2S = 2 + 4 + 8 + 16 + 32 + ... = S - 1$$

Hence $S = -1,$ so that -1 is positive.

9. Here I have proved the wrong statement that, 0 is positive (*i.e.*, greater than zero).
Find the mistake in the following proof.

Proof : Write $u = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...$ and $v = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + ...$

Then $2v = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = u + v$ so that $u - v = 0.$

But, on subtracting corresponding terms,

$$u - v = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + ...$$

where each bracketed terms is greater than zero.

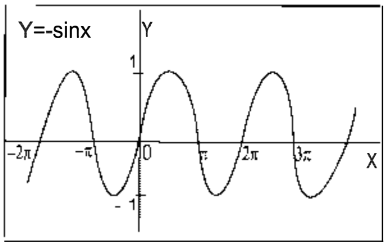
Thus $u - v$ is greater that zero or 0 is greater that zero.

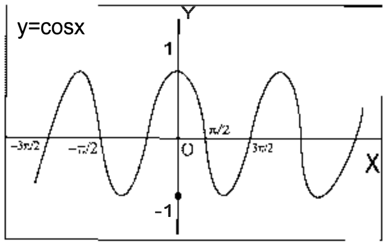
10. Here I have proved the wrong statement that $-2\pi = 0.$ Find the mistake in the following proof.

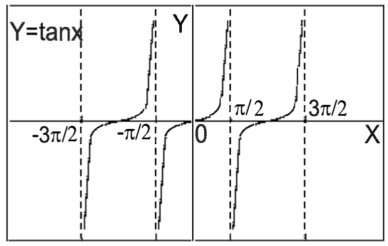
Proof : As we know that $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$ it follows that for any x

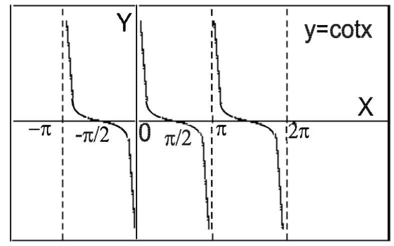
$$e^{ix} = e^{ix}.e^{2xi} = e^{i(x+2x)} \Rightarrow \left(e^{ix}\right)^i = \left(e^{i(x+2\pi)}\right)^i \Rightarrow e^{-x} = e^{-(x+2x)}$$

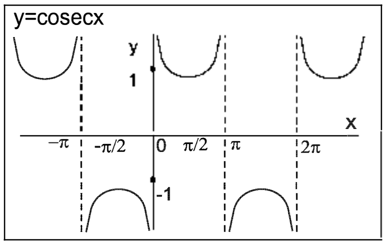
$$\Rightarrow e^{-x} = e^{-x}.e^{-2x} \Rightarrow e^{-2x} = 1 \Rightarrow -2\pi = 0$$

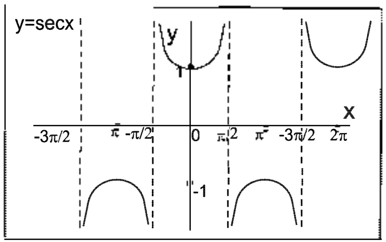

Y=-sinx
Y
X
1
-1
0

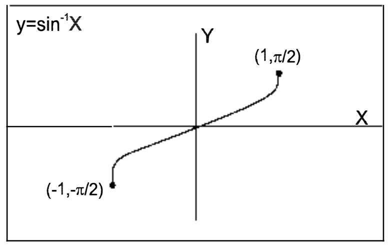

y=cosx
Y
X
1
-1
O

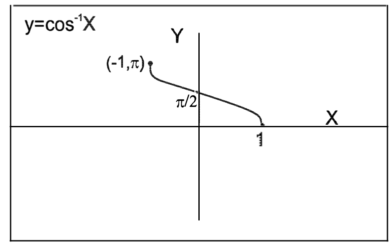

Y=tanx
Y
X
-3π/2
-π/2
0
π/2
3π/2

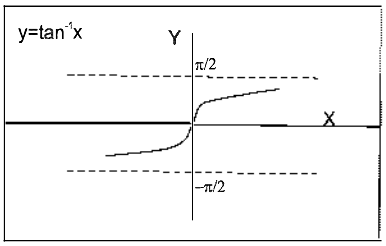

y=cotx
Y
X
-π
-π/2
0
π/2
π
2π

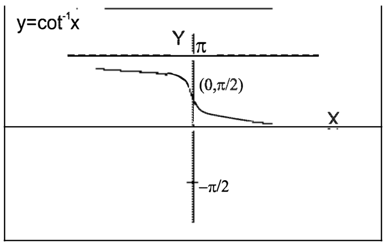

y=cosecx
y
x
1
-1
-π
-π/2
0
π/2
π
2π

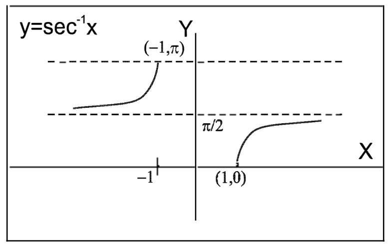

y=secx
y
x
-3π/2
0
-1

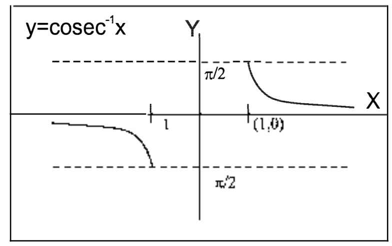

y=sin-1X
Y
X
(1,π/2)
(-1,-π/2)

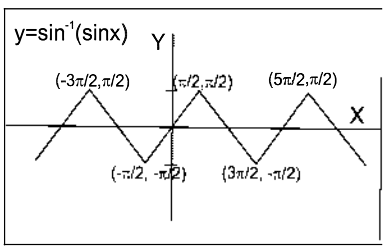

y=cos-1X
Y
X
(-1,π)
π/2
1

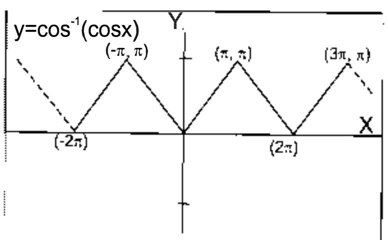

y=tan-1x
Y
X
π/2
-π/2

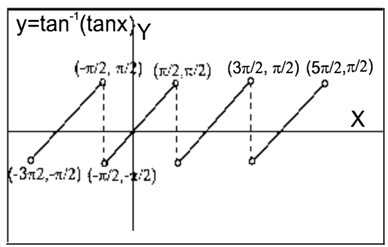

y=cot-1x
Y
π
X
(0,π/2)
-π/2

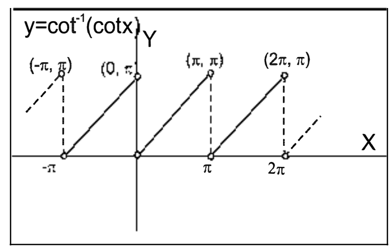

y=sec-1x
Y
X
(-1,π)
π/2
-1
(1,0)

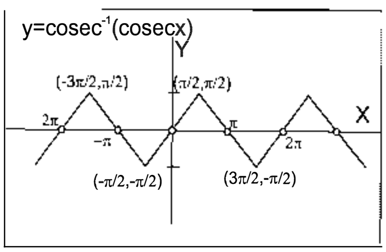

y=cosec-1x
Y
X
π/2
1
(1,0)

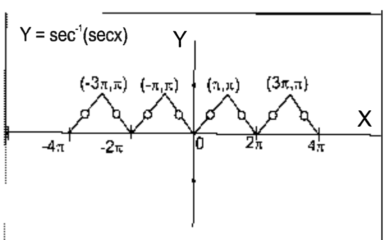

y=sin-1(sinx)
Y
X
(-3π/2,π/2)
(5π/2,π/2)
(3π/2, -π/2)

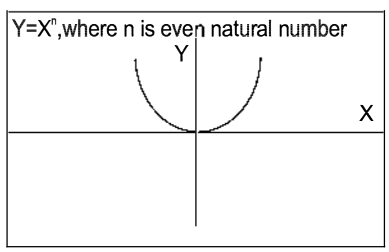

y=cos-1(cosx)
Y
X
(-π, π)
(π, π)
(3π, π)
(-2π)
(2π)

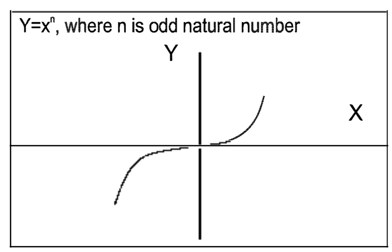

y=tan-1(tanx)
Y
X
(3π/2, π/2)
(5π/2,π/2)

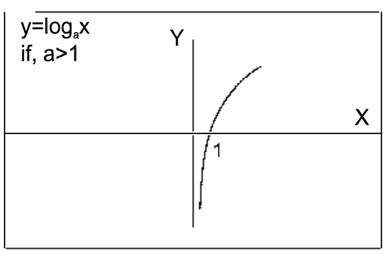

y=cot-1(cotx)
Y
X
(π, π)
(2π, π)
-π
π
2π

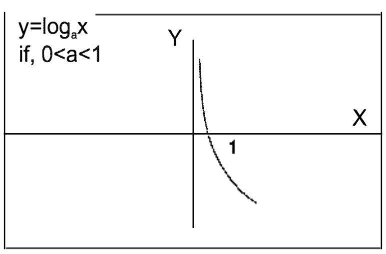

y=cosec-1(cosecx)
Y
X
(-3π/2,π/2)
(-π/2,-π/2)
(3π/2,-π/2)

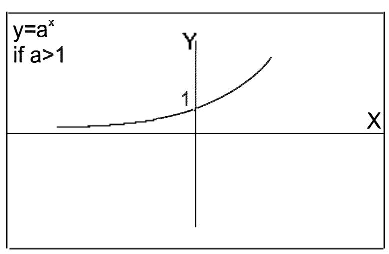

Y = sec-1(secx)
Y
X
(3π,π)
0

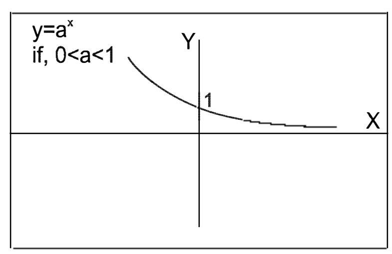

Y=Xn,where n is even natural number
Y
X

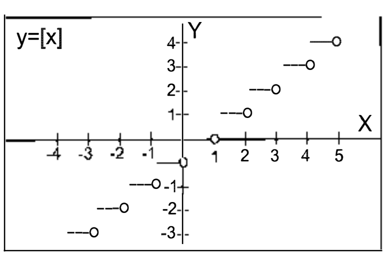

Y=xn, where n is odd natural number
Y
X

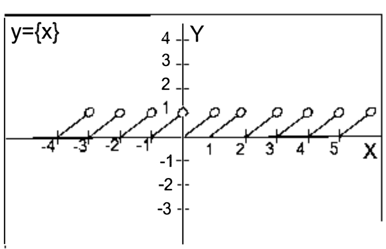

y=logax
if, a>1
Y
X
1


y=logax
if, 0<a<1
Y
X
1


y=ax
if a>1
Y
X
1


y=ax
if, 0<a<1
Y
X
1


y=[x]
Y
X
4
3
2
1
-1
-2
-3
-4
-3
-2
-1
1
2
3
4
5


y={x}
Y
X
4
3
2
1
-1
-2
-3
1
2
3
4
5