CHAPTER 6

APPLICATION OF DERIVATIVES

The sight of soap bubble produced using a bubble wand is very exciting! One application of derivative is finding the rate of increase of size of the bubble (*dv*/*dt*) due to increasing radius, where *V* is the volume of spherical bubble and *r* is the radius. This can be calculated by knowing the rate of increase of radius with time (*dr*/*dt*).

APPLICATION OF DERIVATIVES

Topics to be covered as per C.B.S.E. revised syllabus (2024-25)

- Applications of derivatives:
- rate of change of quantities,
- increasing/decreasing functions,
- maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool).
- Simple problems (that illustrate basic principles and understanding of the subject as well as real life situations).

POINTS TO REMEMBER

- **Rate of change:** Let $y = f(x)$ be a function then the rate of change of y with respect to x is given by $\frac{dy}{dx} = f'(x)$ where a quantity y varies with another quantity x.

  $\left\{\frac{dy}{dx}\right\}_{x = x_1}$ or f' (x_1) represents the rate of change of y w.r.t. x at $x = x_1$.

- **Increasing and Decreasing Function**

  Let f be a real-valued function and let I be any interval in the domain of f. Then f is said to be

  a) Strictly increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

  b) Increasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

  c) Strictly decreasing in I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

  d) Decreasing on I, if for all $x_1, x_2 \in I$, we have

  $$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- **Derivative Test:** Let f be a continuous function on [a, b] and differentiable on (a, b). Then

  a) f is strictly increasing on [a, b] if $f'(x) > 0$ for each $x \in (a, b)$.

  b) f is increasing on [a, b] if $f'(x) \geq 0$ for each $x \in (a, b)$.

  c) f is strictly decreasing on [a, b] if $f'(x) < 0$ for each $x \in (a, b)$.

d) f is decreasing on [a, b] if $f'(x) \leq 0$ for each $x \in (a, b)$.

e) f is constant function on [a, b] if $f'(x) = 0$ for each $x \in (a, b)$.

- **Maxima and Minima**

a) Let f be a function and c be a point in the domain of f such that either f '(x)=0 or f '(x) does not exist are called critical points.

b) **First Derivative Test:** Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in interval I.

i. $f'(x)$ changes sign from positive to negative as x increases through *c*, then *c* is called the point of the local maxima.

ii. $f'(x)$ changes sign from negative to positive as *x* increases through *c*, then *c* is a point of *local minima*.

iii. $f'(x)$ does not change sign as *x* increases through *c*, then *c* is neither a point of *local maxima* nor a point of *local minima*. Such a point is called a point of *inflexion.*

c) **Second Derivative Test :** Let *f* be a function defined on an interval I and let $c \in$ I. Let f be twice differentiable at c. Then

i. $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value $f(c)$ is local maximum value of *f*.

ii. $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value $f(c)$ is local minimum value of *f*.

iii. The test fails if $f'(c) = 0$ and $f''(c) = 0$.

EXTREME VALUE OF A FUNCTION

Let $y = f(x)$ be a real function defined on an interval I and *C* be any point in I. Then *f* is said to have an extreme value in *I* if $f(c)$ is either maximum or minimum value of *f* in *I*.

Here, $f(c)$ is called the extreme value and *C* is called one of the extreme points.

Illustration:

Let $f(x) = (2x - 1)^2 + 3$.

Then, $f(x) \geq 3$, as $(2x - 1)^2 \geq 0$

For any real number 'x'

$\Rightarrow (2x - 1)^2 + 3 \geq 0 + 3$

Thus, minimum value of $f(x)$ is 3, which occurs at $x = \frac{/}{2}$

Also $f(x)$ has no maximum value as $f(x) \to \infty$ as $|x| \to \infty$

Illustration:

Let $g(x) = -(x - 1)^2 + 10$.

Then, $g(x) = 10 - (x - 1)^2 \leq 10 \ \forall x \in R$ as $(x - 1)^2$ is

Always greater them or equal to zero.

Thus maximum value of $g(x)$ is 10, which occurs at $x = 1$

Also $g(x)$ has no minimum value of $f(x) \to -\infty$ as $|x| \to \infty$.

Illustration:

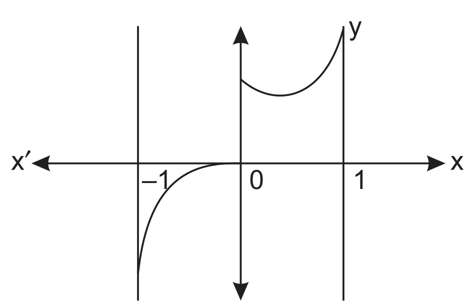
Neither maximum nor minimum value of a function.

Let as consider a function $f(x) = x^3$, $x \in (-1, 1)$

Since this function is an increasing function in (–1, 1), it should have minimum value at a point nearest to –1 and maximum value at a point nearest to 1.

But we can not locate such points (see figure)

So, $f(x) = x^3$, has neither maximum nor-minimum value in (–1, 1).



But, if we extend the domain of f to [–1, 1], then the function $f(x) = x^3$ has maximum value 1 at $x = 1$ and minimum value –1 at $x = -1$

Note: Every continuous function on an closed interval has a maximum and minimum

ONE MARK QUESTIONS

Multiple Choice Questions(MCQ)

1. If a function $f: R \to R$ is defined by $f(x) = 2x + \cos x$, then

   (a) f has a minimum at $x = \pi$

   (b) has a maximum at x=0

   (c) f is a decreasing function

   (d) f is an increasing function

2. If the radius of circle is increasing at the rate of 2cm/sec , then the area of circle when its radius is 20 cm is increasing at the rate of

   (a) $80\pi m^2/sec$ (b) $80\ m^2/sec$

   (c) $80\pi cm^2/sec$ (d) $80\ cm^2/sec$

3. The maximum value of $\frac{\log x}{x}$ is :

   (a) e (b) *2e*

   (c) $\frac{1}{e}$ (d) $\frac{2}{e}$

4. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is :

   (a) $[-l, \infty)$ (b) $(-\infty, -2]$

   (c) $[-2, -l)$ (d) $[-l, 1)$

5. The sides of an equilateral triangle are increasing at the rate of 2cm/sec. The rate at which its area increases, when its side is 10 cm is :

   (a) $10\ cm^2$ / sec (b) $10\sqrt{3}\ cm^2$ / sec

   (c) $\frac{10}{3} cm^2$ / sec (d) $\sqrt{3} cm^2$ / sec

6. The function f(x)= x^x, x > o is increasing on the interval

(a) (0, e] (b) (0, 1/e)

(c) [1/e, ∞) (d) None of these

7. The function f(x) = $2x^3 - 15x^2 + 36x + 6$ is increasing in the interval:

(a) (–∞, 2)U [3, ∞) (b) (–∞, 2)

(c) (–∞, 2]U [3, ∞) (d) [3, ∞)

8. A point on the curve $y^2 = 18x$ at which ordinate increases twice the rate of abscissa is :

(a) (2, 4) (b) (2, –4)

(c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

9. The least value of function $f(x) = ax + \frac{b}{x} (x > 0, a > 0, b > 0)$ is:

(a) $\sqrt{ab}$ (b) $2\sqrt{ab}$

(c) ab (d) 2ab

10. At $x = \frac{5\pi}{6}$, the function f(x) = 2 sin 3x + 3cos 3x is

(a) Maximum (b) Minimum

(c) zero (d) Neither maximum nor minimum

11. The function tanx-x :

(a) always increases (b) always deccreases

(c) Remains contain (d) Sometime increases sometime decreases

12. The minimum value of $x^2 + \frac{250}{x}$ is:

(a) 75 (b) 55

(c) 50 (d) 20

13. In a sphere of radius r, a right circular cone of height having maximum curved surface area is inscribed. The expression for the square of curved surface of the cone is:

(a) $2\pi^2 rh(2rh + h^2)$ (b) $\pi^2 hr(2rh + h^2)$

(c) $2\pi^2 r(2rh^2 - h^3)$ (d) $2\pi^2 r^2(2rh - h^2)$

ASSERTION REASON TYPE QUESTIONS 1 Marks

Statement I is called Assertion (A) Statements II is called Reason R. Read the given statiments carefully and chose the correct answer from the four options given below.

(a) Both the statement are true and statement II is correct explantion of statement I

(b) Both the statmetns are treu and statement II is not the correct explanation of statement I.

(c) Statement I is true statement II is false

(d) Statement I is false and statement II is true

14. Statement I. The function f(x) = x^x, x > 0, is strictly increasing in $\left(\frac{1}{e}, \infty\right)$

Statement II : $\log_a x > b \Rightarrow x > a^b$ if a > 1

15. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + an, x \neq 0$

has extreme values at x = – 1, and x = 2

Statement I : f has local maximum at x = – 1 and x = 2

Statement II : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$

16. Let $f(x) = 2x^3 - 15x^3 + 36x + 1$

Statement I : f is strictly decreasing in [2, 3]

Statement I :: f is strictly increasing in (– ∞, 2] U [3, ∞)

TWO MARKS QUESTIONS

1. The sum of the two numbers is 8, what will be the maximum value of the sum of their reciprocals.
2. Find the maximum value of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3]
3. If the rate of change of Area of a circle is equal to the rate of change its diameter. Find the radius of the circle.
4. The sides of on equilateral triangle are increasing at the rate of 2 cm/s. Find the rate at which the area increases, when side is 10 cm.
5. If there is an error of a% in measuring the edge of cube, then what is the percentge error in its surface?
6. If an error of k% is made in measuring the radius of a sphere, then what is the percentage error in its volume?
7. If the curves $y = 2e^x$ and $y = ae^{-x}$ intersect orthogonally, then find a.
8. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate.
9. Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$.
10. Find the point on the curve $y = x^2$, where the slope of the tangent is equal to the x co-ordinate of the point.
11. Use differentials to approximate the cube root of 66.
12. Find the maximum and minimum values of the function $f(x) = \sin(\sin x)$
13. Find the local maxima and minima of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$.
14. If $y = a \log x + bx^2 + x$ has its exteme values at $x = -1$ and $x = 2$, then find a and b.
15. If the radius of the circle increases from 5 into 5.1 cm, then find the increase in area.

THREE MARKS QUESTIONS

1. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cm^3 of gas per second. Find the rate at which the radius of the balloon is increasing, when its radius is 15 cm.

2. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $\frac{3}{2}$c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

3. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of an edge of the cube.

4. A kite is moving horizontally at a height of 151.5 meters. If the speed of the kite is 10m/sec, how fast is the string being let out when the kite is 250 m away from the boy who is flying the kite ? The height of the boy is 1.5 m.

5. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10-t)^2$. How fast is the water running out at the end of 5 sec. and what is the average rate at which the water flows out during the first 5 seconds?

6. A man 2m tall, walk at a uniform speed of 6km/h away from a lamp post 6m high. Find the rate at which the length of his shadow increases.

7. A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lower most. Its semi- vertical angle is $\tan^{-1}(0.5)$. water is poured into it at a constant rate of $5m^3/h$. Find the rate at which the level of the water is rising at the instant, when the depth of Water in the tank is 4m.

8. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area. Prove that the radius is decreasing at a constant rate.

9. A conical vessel whose height is 10 meters and the radius of whose base is half that of the height is being filled with a liquid at a uniform rate of $1.5m^3/min$. find the rate at which the level of the water in the vessel is rising when it is 3m below the top of the vessel.

10. Let x and y be the sides of two squares such that $y = x - x^2$. Find the rate of change of area of the second square w.r.t. the area of the first square.

11. The length of a rectangle is increasing at the rate of 3.5 cm/sec. and its breadth is decreasing at the rate of 3 cm/sec. Find the rate of change of the area of the rectangle when length is 12 cm and breadth is 8 cm.

12. If the areas of a circle increases at a uniform rate, then prove that the perimeter various inversely as the radius.

13. Show that $f(x) = x^3 - 6x^2 + 18x + 5$ is an increasing function for all $x \in R$. Find its value when the rate of increase of f(x) is least.

[Hint: Rate of increase is least when $f'(x)$ is least.]

14. Determine whether the following function is increasing or decreasing in the given interval: $f(x) = \cos\left(2x + \frac{\pi}{4}\right), \frac{3\pi}{8} \leq x \leq \frac{5\pi}{8}$.

15. Determine for which values of x, the function y=$x^4 - \frac{4x^3}{3}$ is increasing and for which it is decreasing.

16. Find the interval of increasing and decreasing of the function $f(x) = \frac{\log x}{x}$

17. Find the interval of increasing and decreasing of the function $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$.

18. Show that $f(x) = x^2 e^{-x}, 0 \leq x \leq 2$ is increasing in the indicated interval.

19. Prove that the function $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

20. Find the intervals in which the following functionis decreasing.

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

21. Find the interval in which the function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}, x > 0$ is strictly decreasing.

22. Show that the function $f(x) = \tan^{-1}(\sin x + \cos x)$,is strictly increasing the interval $\left(0, \frac{\pi}{4}\right)$.

23. Find the interval in which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is increasing or decreasing.

24. Find the interval in which the function given by

$$f(x) = \frac{3x^4}{10} - \frac{4x^3}{5} - 3x^2 + \frac{36x}{5} + 11$$

(i) strictly increasing

(ii) strictly decreasing

25. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

26. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 Units/sec. then how fast is the slope of the curve changing when x=3?

27. If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

28. If the side of a cube be increased by 0.1%, find the corresponding increase in the volume of the cube.

29. Find the maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

30. Find the absolute maximum value and absolute minimum value of the following question $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$ in [-2, 2.5]

31. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval [0, 1]

32. Find the absolute maximum and absolute minimum value of $f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

33. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

FIVE MARKS QUESTIONS

1. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$.

2. If the sum of length of hypotenuse and a side of a right angled triangle is given, show that area of triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

3. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

4. The sum of the surface areas of cuboids with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x = 3 radius of the sphere. Also find the minimum value of the sum of their volumes.

5. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

6. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere.

7. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

8. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle $\alpha\ is\ \frac{4}{27}\pi h^3 tan^2\alpha.$ Also show that height of the cylinder is $\frac{h}{3}$

9. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2,1).

10. Find the shortest distance between the line $y - x = 1$ and the curve$x = y^2$.

11. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum?

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius r is $\frac{2r}{\sqrt{3}}$.

13. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

SELF ASSESSMENT-1

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of curve changing when $x = 3$

   (a) 72 units/sec (b) –72 units/sec

   (c) 54 units/sec (d) –54 units/sec

2. The function $f(x) = \tan x - 4x$, on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ is

   (a) strictly decreasing (b) strictly increasing

   (c) neither increasing nor deceasing

   (d) None of these

3. The curve $y = xe^x$ has minimum value equal to

(a) 1 (b) 0

(c) –e (d) $/\ \frac{1}{e}$

4. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate (in cm^2/sec) at which the area increases, when side is 10 cm is

(a) 10 (b) 5

(c) $10\sqrt{3}$ (d) $5\sqrt{3}$

5. If $ab = 2a + 3b$, $a > 0$, $b > 0$ then the minimum value of ab is

(a) 6 (b) 12

(c) 24 (d) 48

SELF ASSESSMENT-2

EACH OF THE FOLLOWING MCQ HAS ONE OPTION CORRECT, CHOOSE THE CORRECT ALTERNATIVE.

1. If the function $f(x) = 2x^3 – 9ax^2 + 12a^2 x + 1$ where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a =

(a) 0 (b) 1

(c) 2 (d) 3

2. The interval in which $y = –x^3 + 3x^2 + 2022$ is increasing is

(a) $(–\infty, 0) \cup (2, \infty)$ (b) $(2, \infty)$

(c) $(0, 2)$ (d) $(–\infty, 0)$

3. The maximum value of the function $f(x) = 4\sin x.\cos x$ is

(a) 1 (b) 2

(c) 3 (d) 4

4. Which of the following function is decreasing on $\left(0, \frac{\pi}{2}\right)$

(a) $\cos x$ (b) $\sin x$

(c) $\tan x$ (d) $\sin 2x$

5. A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. The rate at which the length of his shadow increases is

(a) 5 km/hr (b) 2 km/hr

(c) 3 km/hr (d) 2.5 km/hr

Answers

ONE MARK QUESTIONS

Answer

1. (d) f is an increasing
2. (c) $80\pi\ cm^2 / sec$
3. (c) $\frac{1}{e}$
4. (c) [–2, –1]
5. (b) $10\sqrt{3}cm^2 / sec$
6. (c) [1/e, ∞)
7. (c) (–∞, 2] U [3, ∞)
8. (d) $\left(\frac{9}{8}, \frac{7}{2}\right)$
9. (b) $2\sqrt{ab}$
10. (d) Neither maximum nor minimum
11. (a) always increases
12. (a) 75
13. (c) $2\pi^2 r(2rh^2 - h^3)$
14. (a)
15. (a)
16. (b)

TWO MARKS QUESTIONS

1. $\frac{1}{2}$
2. 89
3. $\frac{1}{\pi}$ units
4. $10\sqrt{3}\, cm^2 / s$
5. 2a%
6. 3k %
7. ½
8. (2, 4)
10. (0, 0)
11. 4.042
12. sin 1, – sin 1
13. Local maxima at x = 1
    Local minima at x = 6
14. a = 2, b = –½
15. $\pi\ cm^2$

THREE MARKS QUESTIONS

1. $\frac{1}{\pi}\, cm / s$
2. $\frac{3}{8\pi}\, cm / min$
4. 8 m/sec.
5. 3000 L/s
6. 3 km/h
7. $\frac{35}{88}$ m/h
9. $\frac{6}{49\pi}$ m/min.

10. $1 - 3x + 2x^2$

11. 8 cm^2/sec

13. 25

14. Increasing

15. Increasing for all $x \geq 1$
Decreasing for all $x \leq 1$

16. Increasing on (o, e)

Decreasing on [e, ∞)

17. Increasing on

$$\left(O, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$$

Decreasing on $[\frac{3\pi}{4} . \frac{7\pi}{4}]$

20. $(-\infty, 1]U[2, 3]$

21. [1,∞]

23. increasing on [0, ∞)

Decreasing $(-\infty, 0]$

24. (i) Strictly increasing
[-2,1] U [3, ∞)

(ii) Strictly decreasing
(-∞, -2] U [1, 3]

26. decrease 72 units/sec.

27. $\pi\ cm^2$

28. 0.3%

29. max. value $=\frac{3}{4}$, mim value $=\frac{1}{2}$

30. ab. Max.$= \frac{157}{8}$, ab. Min. $= \frac{-7}{4}$

31. max.value=0,

min.value $= \frac{-3}{5}\left[\frac{2}{5}\right]^{2/3}$

32. ab. Max = 14 at x = 9

ab. Min.$= \frac{-3}{4^{4/3}}$ at $x = \frac{5}{4}$

33. π

FIVE MARKS QUESTIONS

4. $18r^3 + \frac{4}{3}\pi r^3$

9. (1, 2)

10. $\frac{3\sqrt{2}}{8}$

11. $\frac{144}{\pi+4}m, \frac{36\pi}{\pi+4}m$

13. 2ab sq. Units.

SELF ASSESSMENT TEST-1

1. (b) 2. (a) 3. (d) 4. (c) 5. (c)

SELF ASSESSMENT TEST-2

1. (c) 2. (c) 3. (b) 4. (a) 5. (d)